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Himanshu
Gaurav Singh

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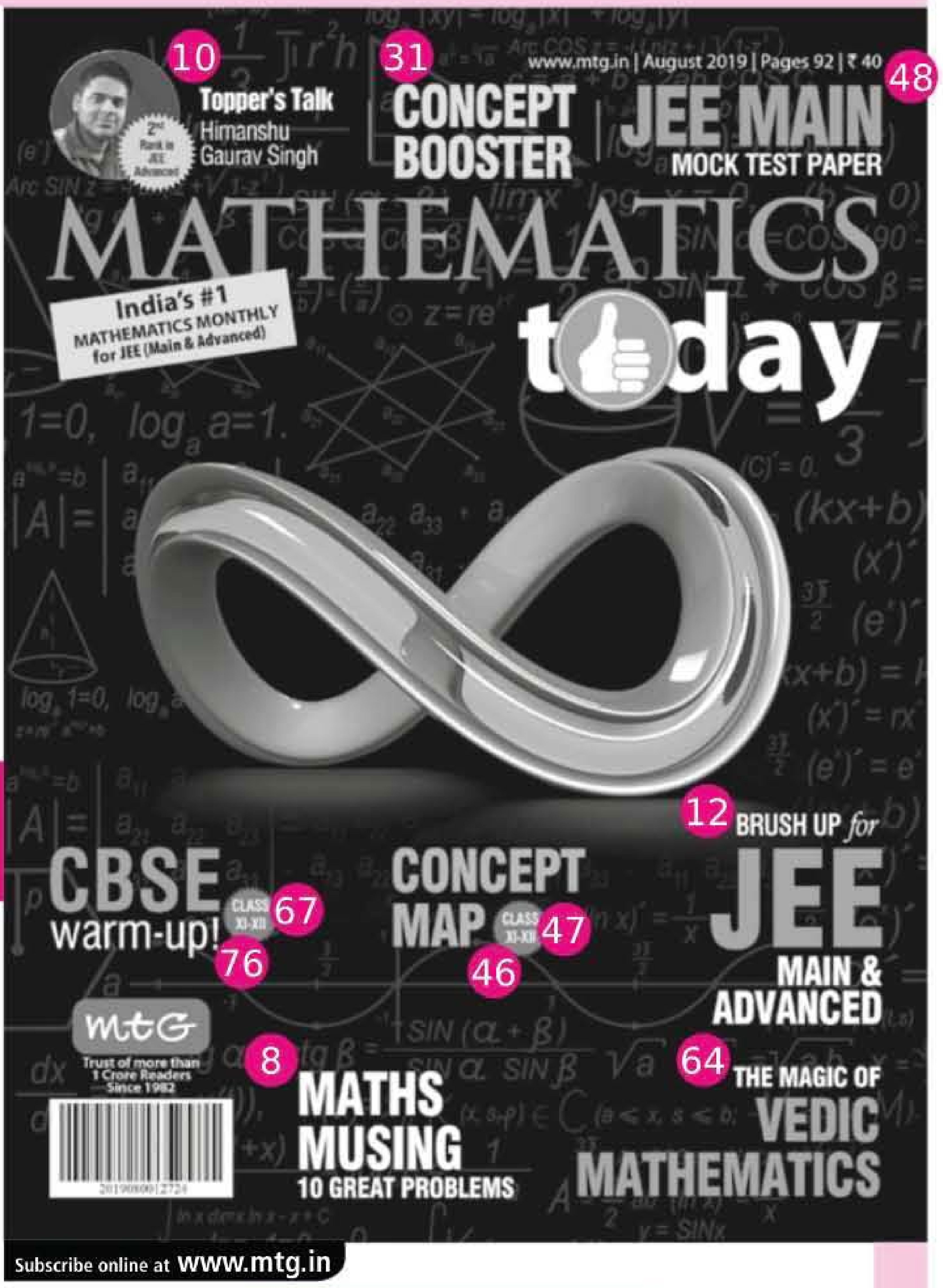
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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

PROBLEM
Set 200

JEE MAIN

1. If $\lim\limits_{x\to 0}(\cos x + a\sin bx)^{1/x} = e^2$, then the values of a and b can be
   (a) $a = 1, b = -2$ (b) $a = 2\sqrt{2}, b = \sqrt{2}$
   (c) $a = 2\sqrt{2}, b = \frac{1}{\sqrt{2}}$ (d) $a = -2, b = 1$

2. If α is the only real root of the equation $x^3 + bx^2 + cx + 1 = 0$ $(b < c)$, then the value of $\tan^{-1}\alpha + \tan^{-1}\left(\frac{1}{\alpha}\right)$ is equal to
   (a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$ (c) 0 (d) $\pi/3$

3. Let $f(x) = 1 + 2x^2 + 2^2x^4 + ... + 2^{10}x^{20}$. Then, $f(x)$ has
   (a) more than one minimum
   (b) exactly one minimum
   (c) at least one maximum
   (d) none of these

4. Let $f(x) = \ln\left[\cos|x| + \frac{1}{2}\right]$, where $[\cdot]$ denotes the greatest integer function, then $\int\limits_{x_1}^{x_2} \lim\limits_{n\to\infty}\left(\frac{\{f(x)\}^n}{x^2+\tan^2 x}\right)dx$, where $x_1,\ x_2 \in \left[\frac{-\pi}{6}, \frac{\pi}{6}\right]$ is equal to
   (a) 1 (b) 2 (c) -1 (d) 0

5. If three vectors $\vec{a}, \vec{b}, \vec{c}$ are such that $\vec{a} \neq \vec{0}$ and $\vec{a}\times\vec{b} = 2(\vec{a}\times\vec{c}), |\vec{a}| = |\vec{c}| = 1, |\vec{b}| = 4$ and the angle between $\vec{b}$ and $\vec{c}$ is $\cos^{-1}\left(\frac{1}{4}\right)$, then $\vec{b} - 2\vec{c} = \lambda\vec{a}$, where λ is equal to
   (a) ± 4 (b) -2 (c) ± 3 (d) 2

JEE ADVANCED

6. If the term independent of x in the expansion of $\left(\sqrt{x} - \lambda / x^2\right)^{10}$ is 405, then the value of λ is
   (a) -3 (b) 9 (c) -9 (d) 3

COMPREHENSION

If $(f(x))^2 f\left(\frac{1-x}{1+x}\right) = 64x\ \forall x \in D_f$, then

7. $f(x)$ is equal to
   (a) $4x^{2/3}\left(\frac{1+x}{1-x}\right)^{1/3}$ (b) $x^{1/3}\left(\frac{1-x}{1+x}\right)^{1/3}$
   (c) $x^{2/3}\left(\frac{1-x}{1+x}\right)^{1/3}$ (d) $x\left(\frac{1+x}{1-x}\right)^{1/3}$

8. The value of $f(9/7)$ is
   (a) $8(7/9)^{2/3}$ (b) $4(9/7)^{1/3}$
   (c) $-8(9/7)^{2/3}$ (d) none of these

NUMERICAL ANSWER TYPE

9. If a, b, c are in H.P. and if $\left(\frac{a+b}{2a-b}\right) + \left(\frac{c+b}{2c-b}\right) > \sqrt{\lambda\sqrt{\lambda\sqrt{\lambda ...\infty}}}$, then the value of λ must be

MATRIX MATCH

10. Match the columns :

	Column-I		Column-II
P.	If $\lvert z+\bar{z}\rvert + \lvert z-\bar{z}\rvert = 2$, then z lies on	1.	pair of straight lines
Q.	If $\lvert z+\bar{z}\rvert = \lvert z-\bar{z}\rvert$, then z lies on	2.	circle
R.	If $\lvert z\rvert = 1$, then $2/z$ lies on	3.	square

	P	Q	R		P	Q	R
(a)	3	2	1	(b)	2	1	3
(c)	3	1	2	(d)	2	3	1

See Solution Set of Maths Musing 199 on page no. 85

AIR
2

Himanshu Gaurav Singh

Cracking the JEE Advanced EXAM

TOPPER talks to you...

MTG : Why did you appear for Engineering Entrance?
Himanshu : Since childhood, I have been interested in Mathematics and Science which eventually led to interest in Engineering.

MTG : What exams have you appeared for and what is your rank in these exams?
Himanshu : I have appeared for JEE Main and JEE Advanced securing AIR 14th and AIR 2nd respectively.

MTG : How many hours in a day did you study to prepare for the examination?
Himanshu : I used to study 4 to 6 hours a day.

MTG : How much time does one require for serious preparation for this exam?
Himanshu : According to me two years' dedicated preparation is sufficient to crack such exams.

MTG : Any extra coaching?
Himanshu : I joined FIITJEE since Class-X.

MTG : On which topic and chapters you laid more stress in each subject?
Himanshu : I gave equal attention to each and every topic in all subjects.

MTG : Which subject/topic you were strong/weak at?
Himanshu : I was equally strong in all three subjects.

MTG : Which books/Magazines/Coaching classes you followed?
Himanshu : Study material provided by my coaching was my main focus. FIITJEE provided a lot of mock tests. There were weekly tests, monthly tests and more. So, that kept me always in touch with all topics of all subjects.

MTG : In your words what are the components of an ideal preparation plan?
Himanshu : Consistency, hard work and most importantly the support of my parents and teachers proved to be key factors in my preparation.

MTG : What role did the following play in your success:
Himanshu : Parents : My parents provided me an ideal study environment since childhood and always supported me.
Teachers : I would like to thank Mr. R K Thakur (my Maths teacher), Srikant Sir (my Physics teacher) and Mayank Sir (my Chemistry teacher) for helping me in becoming strong in concepts and develop a better temperament.
School : My school provided me enough liberty to pursue personal endeavours in academics apart from boards.

MTG : Your family background?
Himanshu : My father is a professor of Electronic Engineering in Govt. Polytechnic U.P. My mother is a homemaker. My younger brother is in Class-X.

MTG : What mistakes you think you shouldn't have made?
Himanshu : I should have slept properly on the night before JEE Advanced 2019 exam.

MTG : Was this your first attempt?
Himanshu : Yes, it was my first attempt.

MTG : What do you think is the secret of your success?
Himanshu : Consistency in hard work is the secret of my success.

MTG : Had you not been selected then what would have been your future plan?
Himanshu : I was quite sure of my selection.

MTG : What advice would you like to give our readers who are JEE aspirants?
Himanshu : Good students should be given more freedom from general activities so that they can tap their true potential. You get, what you deserve, eventually with hard work and dedication.
Be consistent throughout the two years of your preparation. Be calm and stress-free.

☺☺

BRUSH UP for JEE Main & Advanced 2020

Brush up your concepts to get high rank in JEE (Main and Advanced) by reading this column. This specially designed column is updated year after year according to the requirements of these Entrance Tests.

Unit 2

Quadratic Equations & Complex Numbers

QUADRATIC EQUATIONS

A polynomial equation of the form $ax^2 + bx + c = 0$ $\forall\ a, b, c \in R, a \neq 0$ is called a quadratic equation.

Let α, β are the roots of the equation, then value of α and β are given by

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a},\ \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

If $\Delta = b^2 - 4ac$, then $\alpha = \frac{-b+\sqrt{\Delta}}{2a}$, $\beta = \frac{-b-\sqrt{\Delta}}{2a}$

Nature of Roots

The nature of roots of the quadratic equation

$$ax^2 + bx + c = 0 \quad \ldots (*), \quad (a \neq 0;\ a, b, c \in R)$$

depends upon Δ (defined above).

- If $\Delta < 0$ *i.e.*, $b^2 - 4ac < 0$, then the roots of the equation (*) are non-real complex roots.
- Complex roots always appear in pairs and they will be conjugate of each other. These roots are given by
$$\alpha = \frac{-b+i\sqrt{|\Delta|}}{2a},\ \beta = \frac{-b-i\sqrt{|\Delta|}}{2a}$$
- If $\Delta = 0$, then $b^2 = 4ac$ and $\alpha = \beta = \frac{-b}{2a}$ *i.e.*, roots of equation (*) are real and equal and the expression $ax^2 + bx + c$ can be expressed as $a(x-\alpha)^2$.
- If $\Delta > 0$, then $b^2 > 4ac$ *i.e.*, roots of the equation (*) are real and distinct, say α and β, then $ax^2 + bx + c$ can be expressed as $a(x-\alpha)(x-\beta)$.
- If $a, b, c \in Q$ and Δ is a perfect square, then roots of the equation (*) are rational.
- If $a, b, c \in Q$ and Δ is not a perfect square, then roots of the equation (*) are irrational and conjugate of each other. If one root is $p+\sqrt{q}$, then other root is $p-\sqrt{q}$ and vice versa.
- If the quadratic equation (*) has more than two roots (real or complex), then it becomes an identity *i.e.*, $a = b = c = 0$.

Relationship between the roots and coefficients of a quadratic equation

Let α, β are the roots of quadratic equation $ax^2 + bx + c = 0$, $\forall\ a \neq 0$, $a, b, c \in R$, then

- Sum of roots $= \alpha + \beta = \frac{-b}{a} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } x^2}$
- Product of roots $= \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$
- The quadratic equation whose roots are α and β is $(x-\alpha)(x-\beta) = 0$ *i.e.*, $x^2 - (\alpha+\beta)x + \alpha\beta = 0$ *i.e.*, x^2 – (sum of roots)x + product of roots = 0.

Relationship between the roots and coefficients of a polynomial equation of higher degree

Let $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$ ($a \neq 0$, $a_1, a_2, \ldots, a_n$ are constants) be a polynomial equation of degree n. This equation can have n and only n roots. Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be its n roots, then

- Sum of roots (*i.e.*, $\alpha_1 + \alpha_2 + \ldots + \alpha_n$)
$$= (-1)^1 \frac{a_1}{a_0} = (-1)\frac{\text{Coefficient of } x^{n-1}}{\text{Coefficient of } x^n}$$

By : **R. K. Tyagi**, Retd. Principal, HOD Maths, Samarth Shiksha Samiti, New Delhi

- Sum of the products of roots taken two at a time

$$=\sum \alpha_1\alpha_2=(-1)^2\frac{a_2}{a_0}=(-1)^2\frac{\text{Coefficient of } x^{n-2}}{\text{Coefficient of } x^n}$$

- Sum of the products of roots taken three at a time

$$=\sum \alpha_1\alpha_2\alpha_3=(-1)^3\frac{a_3}{a_0}=(-1)^3\frac{\text{Coefficient of } x^{n-3}}{\text{Coefficient of } x^n}$$

- In general, sum of products of the roots taken all at a time *i.e.*, $\alpha_1\alpha_2\alpha_3 \ldots \alpha_n=\prod_{i=1}^{n}\alpha_i=(-1)^n\frac{a_n}{a_0}$

$$=(-1)^n\frac{\text{Constant term}}{\text{Coefficient of } x^n}$$

Condition for Common Roots of Quadratic Equation

Let two quadratic equations are

$$ax^2+bx+c=0 \quad \ldots\text{(i)}$$

and $a'x^2+b'x+c'=0$ ...(ii)

(where $a, a' \neq 0$ and $a'b \neq ab'$)

Let α be a common root of equation (i) and (ii), then

$$a\alpha^2+b\alpha+c=0 \text{ and } a'\alpha^2+b'\alpha+c'=0$$

Solving these equations, we have

$$\frac{\alpha^2}{bc'-b'c}=\frac{-\alpha}{c'a-ca'}=\frac{1}{ab'-ba'}$$

$$\Rightarrow \alpha^2=\frac{bc'-b'c}{ab'-ba'} \text{ and } \alpha=\frac{ca'-c'a}{ab'-ba'}$$

Now, by eliminating α, we get

$$(bc'-b'c)(ab'-a'b)=(ca'-c'a)^2,$$

which is the required condition for one root of two quadratic equations to be common.

- If $ab'-a'b=0$, then $ca'-c'a=0 \Rightarrow \frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$, which is the condition for both roots of two quadratic equations to be identical (equal/symmetrical).

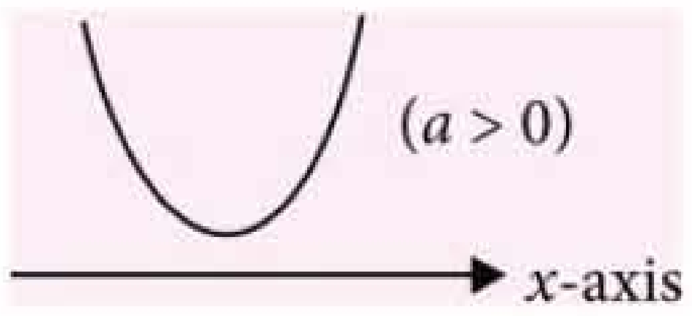
Graph of a Quadratic Polynomial Expression

Let $y=f(x)=ax^2+bx+c$ $(a \neq 0,\ a, b, c \in R)$ be a quadratic polynomial.

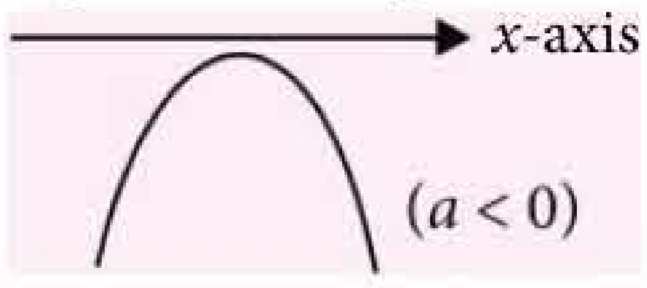
$$\Rightarrow y=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{\Delta}{4a^2}\right] \Rightarrow \left(y+\frac{\Delta}{4a}\right)=a\left(x+\frac{b}{2a}\right)^2$$

Let $Y=y+\frac{\Delta}{4a}$ and $X=x+\frac{b}{2a}$, then we get $Y=aX^2$ or $X^2=\frac{Y}{a}$, which is a parabola.

(a) If $a>0$, then parabola opens upwards.



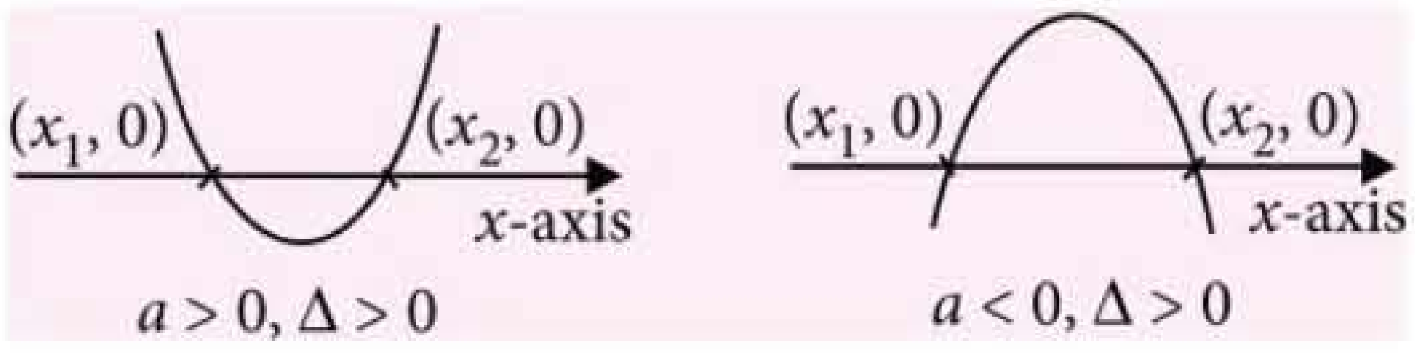
(b) If $a<0$, then parabola opens downwards.



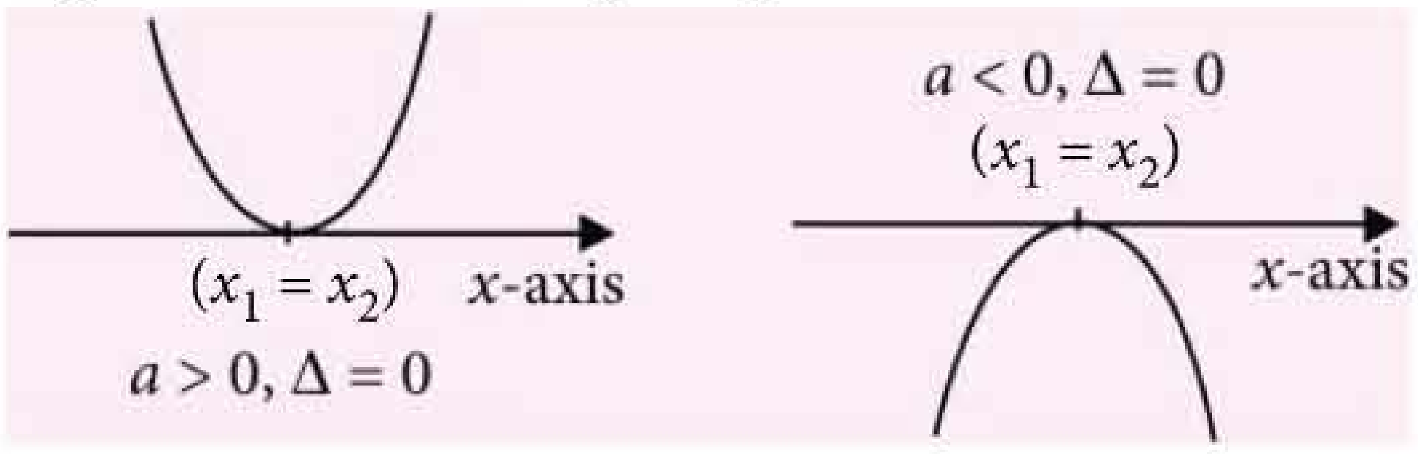
Point of Intersection with Axis

- **Intersection with *x*-axis :** To find the point of intersection of parabola and x-axis, we take $y=0$ *i.e.*, $ax^2+bx+c=0$ and solve this equation for values of x, which are given by $x=\frac{-b\pm\sqrt{\Delta}}{2a}$

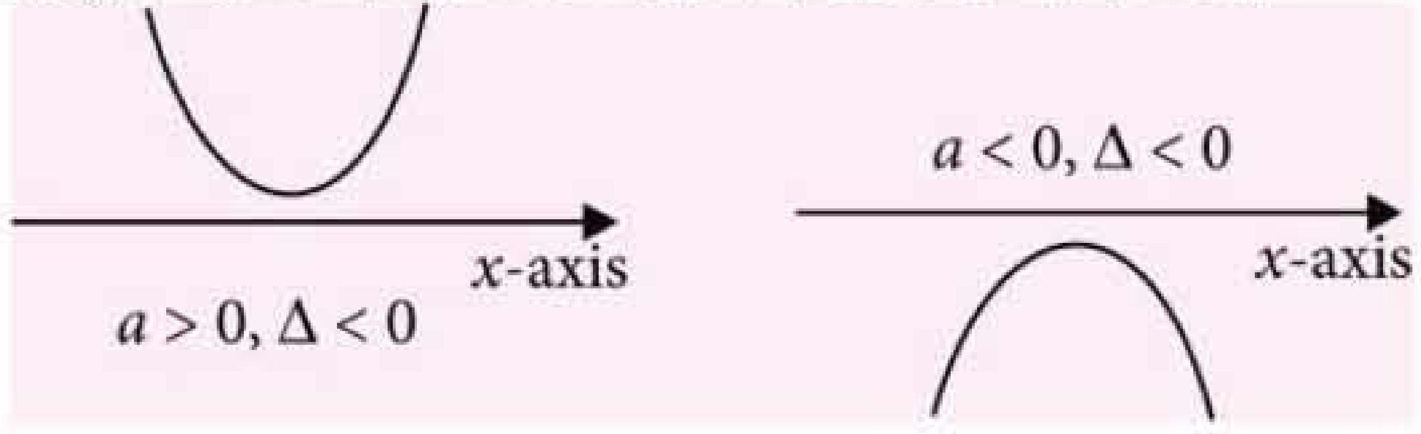
  Case (i) : For $\Delta>0$, parabola cuts x-axis in two distinct points x_1 and x_2.



  Case (ii) : For $\Delta=0$, parabola touches x-axis in one point at which $x_1=x_2$.



  Case (iii) : For $\Delta<0$, parabola does not meet (cuts) x-axis and no real values of x exist.



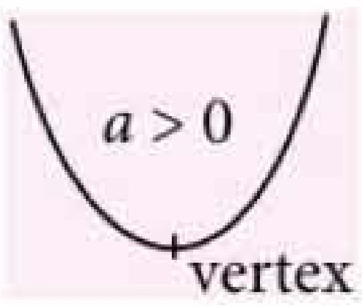
- **Intersection with *y*-axis :** To find the point of intersection of parabola and y-axis, we take $x=0$ in $y=ax^2+bx+c$ and we get $y=c$. Thus the point of intersection of parabola and y-axis is $(0, c)$.
- Greatest and least value or maximum and minimum value of $f(x)=y=ax^2+bx+c$ $(a \neq 0)$
  Vertex of $y=ax^2+bx+c$ or $Y=aX^2$ can be obtained by putting $X=0$, $Y=0$.

$$\Rightarrow x=-\frac{b}{2a},\ y=\frac{-\Delta}{4a}$$

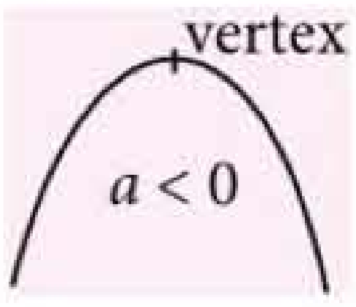
$$\therefore \text{ Vertex of parabola is } \left(-\frac{b}{2a}, \frac{-\Delta}{4a}\right).$$

- For $a>0$, $f(x)$ has least value at

$$x=-\frac{b}{2a}$$



$\therefore$ Minimum/least value of $f(x)$ is $f\left(-\frac{b}{2a}\right)=\frac{-\Delta}{4a}$



- For $a < 0$, $f(x)$ has greatest/maximum value at $x=\frac{-b}{2a}$

  $\therefore$ Greatest/largest/maximum value of $f(x)$ is $f\left(\frac{-b}{2a}\right)=-\frac{\Delta}{4a}$

  Note : $y = f(x) = ax^2 + bx + c > 0 \ \forall \ x \in R$
  $\Leftrightarrow \ a > 0$ and $\Delta < 0$ and
  $ax^2 + bx + c < 0 \ \forall \ x \in R \ \Leftrightarrow a < 0$ and $\Delta < 0$

Maximum and Minimum Value of Rational Quadratic Polynomials

If $f(x)$ is of the form $f(x)=\frac{g(x)}{p(x)}$, where $g(x) = ax^2 + bx + c \ (a \neq 0)$ and $p(x) = Ax^2 + Bx + K \neq 0$, $A \neq 0$, then to find the range of $f(x)$ we proceed the following steps :

Step-1 : Let $l = B^2 - 4AK$, $m = 4aK + 4Ac - 2bB$, $n = b^2 - 4ac$.

Step-2 : Find $l : m : n$ and express the ratio in simplest form.

Step-3 : If $l < 0$, then range of $f(x)$ will be the solution of inequation (inequality) $lx^2 + mx + n \geq 0$.

Step-4 : If $l > 0$, then range of $f(x)$ will be the solution of inequation $lx^2 + mx + n \leq 0$.

Transformation of Equations

Let $ax^2 + bx + c = 0$ $(a \neq 0,\ a,\ b,\ c \in R)$ be a given quadratic equation (the equation may be polynomial equation of degree ≥ 3) and we required or need new equation whose roots are negative and reciprocal of the roots of original equation, then to obtain such new equation replace x by $(-1/x)$ in original equation. Therefore, the equation whose roots are negative and reciprocal of the roots of $ax^2 + bx + c = 0$ is given by

$$a\left(-\frac{1}{x}\right)^2+b\left(\frac{-1}{x}\right)+c=0 \quad i.e.\ cx^2-bx+a=0$$

Thus the roots of $f(x) = 0$ and $f(-1/x) = 0$ are negative and reciprocal of each other.

- If the roots of new equation are multiple or submultiple of original equation

  $$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a^n = 0, \quad ...\ (A)$$

  then to obtain the new equation, each term of original equation is multiplied by multiple or submultiple of suitable power. If roots of new equation are k times the roots of equation (A), then new equation with respect to (A) is obtained by multiplying the terms by k^0, k^1, k^2, ..., k^n from left to right when equation (A) is written in decreasing power of the variable and no term is missing in descending power of variable. If any term is missing then write term as $0 \times$ suitable power of the variable.
- If the roots of a new equation are squares of the roots of the original equation, then to obtain new equation replace x by $x^{1/2}$ in the original equation and if roots of new equation are cubes of the original equation then to get new equation replace x by $x^{1/3}$ in the original equation.
- If roots of new equation are one more than the roots of original equation, then replace x by $x - 1$ in original equation to obtain new equation.
- If roots of a new equation are one less than the roots of original equation then to get new equation replace x by $x + 1$ in original equation.
- The roots of $f(x) = 0$ and $f(-x) = 0$ are equal in magnitude but opposite in sign.
  i.e., If α and β are roots of $ax^2 + bx + c = 0$ then roots of $a(-x)^2 + b(-x) + c = 0$ are $-\alpha$, $-\beta$ thus roots of $f(x) = 0$ and $f(-x) = 0$ are opposite in sign but equal in magnitude.

Important Points about Quadratic Equations

- If α and β are the roots of a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0,\ a,\ b,\ c \in R)$, then equation
  - (a) whose roots are $\alpha \pm k$, $\beta \pm k$ is
    $$a(x \mp k)^2 + b(x \mp k) + c = 0$$
  - (b) whose roots are $\frac{\alpha}{k}, \frac{\beta}{k}$ is $ak^2x^2 + bkx + c = 0$
  - (c) whose roots are $k\alpha$, $k\beta$ is $ax^2 + kbx + k^2c = 0$
  - (d) whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$ is $cx^2 + bx + a = 0$
- If in a quadratic expression/function, $f(x) = ax^2 + bx + c$, $a > 0$, $\Delta = b^2 - 4ac < 0$ then expression $f(x)$ is positive *i.e.*, $f(x) > 0$.
- If in a quadratic expression/function, $f(x) = ax^2 + bx + c$, $a < 0$ and $\Delta = b^2 - 4ac < 0$, then value of $f(x)$ is always negative *i.e.* $f(x) < 0$.
- If sum of the coefficients of polynomial is zero, then $x = 1$ is always a root of $f(x) = 0$.
- Consider quadratic equation $f(x) = ax^2 + bx + c = 0$, $(a \neq 0,\ a,\ b,\ c \in R)$.
  - (a) If $\Delta \geq 0$ and sign of a and c are opposite of sign of b, then both roots of the equation are positive *i.e.*, > 0.

(b) If $\Delta \geq 0$ and a, b, c are all of same sign, then roots of $f(x) = 0$ are negative *i.e.*, < 0.

(c) If $a > 0$, $b = 0$ and $c < 0$, then roots of $f(x) = 0$ are irrational.

(d) If $a > 0$, $b = 0$ and $c > 0$, then roots of $f(x) = 0$ will be imaginary. The modulus values of these roots will be same.

- Let $f(x) = ax^2 + bx + c$, $a \neq 0$, $a, b, c \in R$ and $f(1) < 0$ *i.e.*, $a + b + c < 0$ and equation $f(x) = 0$ has no real roots then c will be < 0.
- If both roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0, a, b, c \in R)$ are greater than unity, then $f(x) = ax^2 + bx + c \Rightarrow f(1) > 0$ *i.e.*, $a + b + c > 0$.
- If $a, b, c > 0$ then equation $a|x|^2 + b|x| + c = 0$ have no real roots, which means both roots will be imaginary and conjugate of each other.
- Let a, b, c are consecutive integers then polynomial equation $f(x) = (x - a)^2 + (x - b)^2 + (x - c)^2 = 0$ have no real roots.

  (a) In this case roots will be imaginary and conjugate of each other.

  (b) $f(x) = (x - a)^2 + (x - b)^2 + (x - c)^2 > 0$.
- If in a cubic equation $ax^3 + bx^2 + cx + d = 0$ $(a \neq 0)$, two roots are equal in magnitude but opposite in sign then its third root must be $= \dfrac{-\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$ and the condition that roots are equal but opposite in sign is $a \times d = b \times c$.
- In any cubic equation of the form $(x - a)^3 + (x - b)^3 + (x - c)^3 = 0$, where a, b, c are consecutive integer, then equation has one real root and two imaginary roots as $f'(x) = 3[(x - a)^2 + (x - b)^2 + (x - c)^2] > 0$ so have no real roots.
- If a cubic equation $ax^3 + bx^2 + cx + d = 0$ has all roots equal, it means roots are in A.P. and G.P. both as sequence of identical numbers satisfies the condition of A.P. and G.P. both and condition for identical/equal roots is $b^3 d = ac^3$.

COMPLEX NUMBERS

Any number which can be expressed in the form $z = x + iy$ $(x, y \in R,\ i = \sqrt{-1})$ is called a complex number, where $x = \text{Re } z$ (real part of z), $y = \text{Im } z$ (imaginary part of z)

- $y = 0$, then $z = x$ is purely real and if $x = 0$, then $z = iy$ is purely imaginary.
- $x = y = 0$, then $z = 0 = 0 + 0i$ is both purely real and purely imaginary.

Algebra of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

- $z_1 = z_2$ if and only if $x_1 = x_2$, $y_1 = y_2$
- $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
- $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$
- $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$
- $\dfrac{z_1}{z_2} = \dfrac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - y_2 x_1)}{x_2^2 + y_2^2}, z_2 \neq 0$

Conjugate of a Complex Number

Conjugate $\bar{z}$: If $z = x + iy$, then its conjugate is $\bar{z} = \overline{x + iy} = x - iy$

- $z\bar{z} = x^2 + y^2$
- $\text{Re}(\bar{z}) = \text{Re}(z) = \dfrac{z + \bar{z}}{2} = x$
- $(\overline{z_1 \pm z_2}) = \bar{z}_1 \pm \bar{z}_2$
- $\overline{z_1 + z_2 + z_3 + ... + z_n} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 + ... + \bar{z}_n$
- $(\overline{z_1 z_2}) = (\bar{z}_1)(\bar{z}_2)$
- $(\overline{z_1\ z_2 .... z_n}) = \bar{z}_1\ \bar{z}_2\ \bar{z}_3\ ....\ \bar{z}_n$
- $\overline{(\bar{z})} = z$
- $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2\,\text{Re}(\bar{z}_1 z_2) = 2\,\text{Re}(z_1 \bar{z}_2)$
- $\overline{(z^n)} = (\bar{z})^n$

If $P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$, where a_1, a_2, a_3 ...., a_n and z are complex numbers, then

$\overline{P(z)} = \bar{a}_0 + \bar{a}_1 \bar{z} + \bar{a}_2 (\bar{z})^2 + \bar{a}_3 (\bar{z})^3 + ... + \bar{a}_n (\bar{z})^n = \bar{P}(\bar{z})$,

where $\bar{P}(z) = \bar{a}_0 + \bar{a}_1 z + \bar{a}_2 z^2 + ... + \bar{a}_n z^n$

- If $R(z) = \dfrac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials in z, $Q(z) \neq 0$, then $\overline{R(z)} = \dfrac{\bar{P}(\bar{z})}{\bar{Q}(\bar{z})}$
- If $z = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$, then $(\bar{z}) = \begin{vmatrix} \bar{x}_1 & \bar{y}_1 & \bar{z}_1 \\ \bar{x}_2 & \bar{y}_2 & \bar{z}_2 \\ \bar{x}_3 & \bar{y}_3 & \bar{z}_3 \end{vmatrix}$

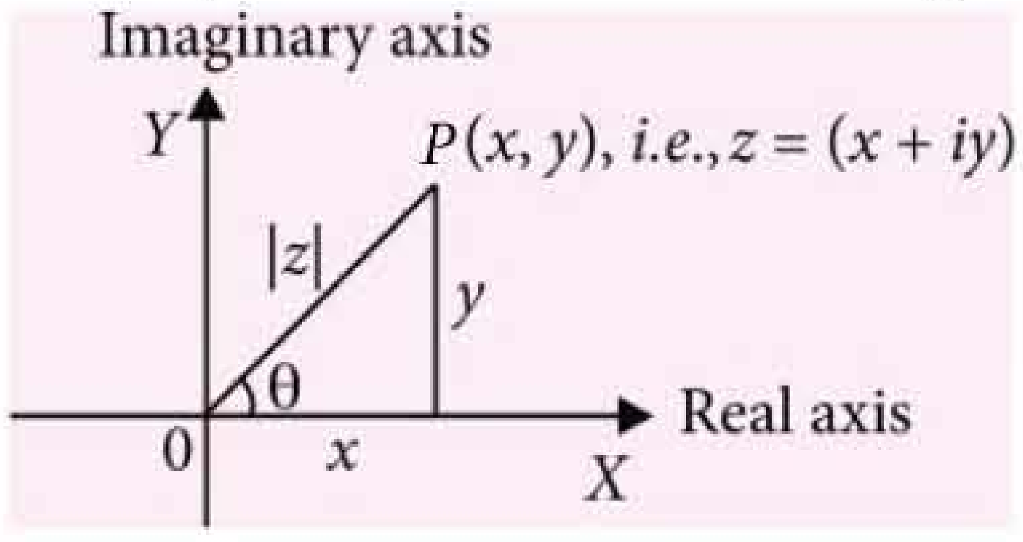
Integral Powers of *i*

- $(i)^0 = 1$, $(i)^1 = i$, $i^2 = -1$, $(i)^3 = i^2 \cdot i = -i$,
  $i^4 = (i^2)^2 = (-1)^2 = 1$
- $(i)^{4m} = 1$ $(\forall\ m \in N)$, $(i)^{4m+1} = (i)^{4m} \cdot i = i$,
  $(i)^{4m+2} = i^2 = -1$, $(i)^{4m+3} = i^3 = -i$

Geometrical Representation

- There is a 1-1 correspondence between the set of complex numbers and the point in x-y plane.
- A complex number $z = x + iy$ corresponds to the point (x, y), $\bar{z}$ to $(x, -y)$, $-z$ to $(-x, -y)$ and $-\bar{z}$ to $(-x, y)$ respectively.

- The number $z = 0$ represent the origin and corresponds to the point $(0, 0)$.
- The real number lies on x-axis called real axis.
- The purely imaginary number lies on y-axis called imaginary axis.
- The xy-plane is called complex or argand plane.
- Geometrical representation of $z = x + iy$

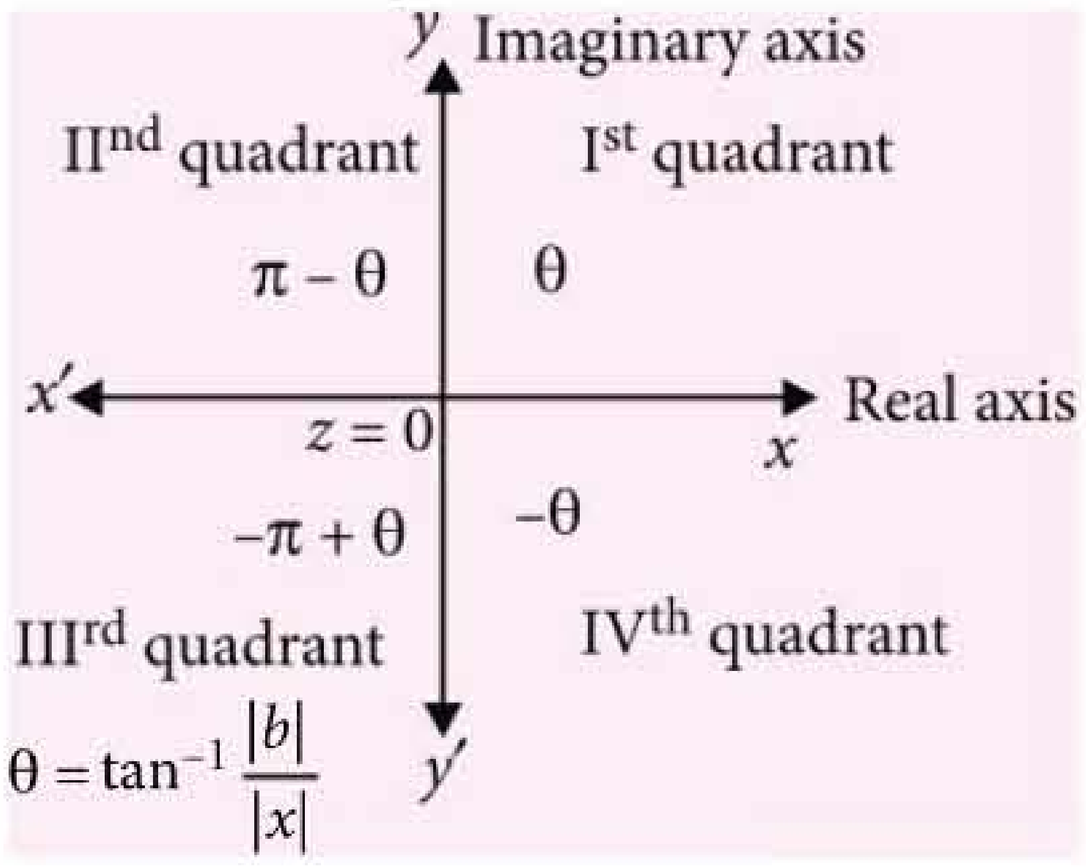


- $OP = |z|$ (modulus of z) $= \sqrt{x^2+y^2}$ = distance of the point z from origin and $\tan\theta = y/x$ or $\theta = \tan^{-1}(y/x)$

  $= \arg(z)$ (argument of z)

  = inclination of OP with positive x-axis

  Principal argument of $z = x + iy = P(x, y)$ when z lies in different quadrants.



Note :

(i) The value of θ such that $-\pi < \theta \le \pi$ is called principal argument of z.

(ii) Order relation for two complex numbers is not defined.

Polar/Trigonometrical and Euler Form of Complex Number

For a complex number, $z = x + iy$, let $x = r\cos\theta$, $y = r\sin\theta$,

$r = \sqrt{x^2+y^2} = |z|$ (modulus of z)

$\theta = \tan^{-1}(y/x) = \arg(z)$, then $z = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$ is called polar or trigonometrical form and $z = re^{i\theta}$ is called exponential or euler form of z, where $e^{i\theta} = \cos\theta + i\sin\theta$

- If $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$; $\frac{1}{z} = \frac{1}{r}e^{-i\theta}$ and $\frac{1}{\bar{z}} = \frac{1}{r}e^{i\theta}$

Properties of Modulus of Complex Number

- $|z|^2 = z\bar{z}$
- $|z| = |-z| = |\bar{z}| = |-\bar{z}|$
- $\mathrm{Re}\, z \le |z|$, $\mathrm{Im}\, z \le |z|$
- $|z| \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z| \le \sqrt{2}|z|$
- $|z_1 z_2 z_3 \ldots z_n| = |z_1||z_2||z_3|\ldots|z_n| = \prod_{j=1}^{n}|z_j|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, $(z_2 \ne 0)$
- **Parallelogram law :**

  $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

Properties of Argument of Complex Number

Let $z = x + iy$ and $n \in I$, then

- $\arg(-z) = \arg(z) + (2n + 1)\pi$
- $\arg(z + \bar{z}) = 0, \pi$ where z is not purely imaginary.
- $\arg(\bar{z}) = \arg\left(\frac{z\bar{z}}{z}\right) = \arg\left(\frac{1}{z}\right) = 2n\pi - \arg z$
- $\arg(z - \bar{z}) = \frac{-\pi}{2}, \frac{\pi}{2}$, where z not being purely real.
- $\arg(z) + \arg(\bar{z}) = 2n\pi$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2n\pi$
- $\arg(z_1 z_2 \ldots z_n) = \arg(z_1) + \arg(z_2) + \ldots + \arg(z_n) + 2n\pi$
- $\arg\left(\frac{z_1}{z_2}\right) = 2n\pi + \arg(z_1) - \arg(z_2) = 2n\pi + \arg(z_1\bar{z}_2)$
- If $\arg\left(\frac{z_1}{z_2}\right) = \theta$, then $\arg\left(\frac{z_2}{z_1}\right) = 2n\pi - \theta$
- $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2n\pi$
- $\arg(z^k) = k\arg(z) + 2n\pi$

Note :

(i) Value of n should be chosen in such a way that R.H.S. of above lies in $(-\pi, \pi]$.

(ii) Argument of a complex number is not unique. If θ be the argument of z, then $2n\pi + \theta$ is also the $\arg(z)$.

(iii) Argument of $z = 0$ cannot be defined.

Triangle Inequality

- $||z_1| - |z_2|| \le |z_1 \pm z_2| \le |z_1| + |z_2|$

(i) $|z_1 + z_2| = |z_1| + |z_2| \Rightarrow \arg(z_1) - \arg(z_2) = 0$

$\Rightarrow \frac{z_1}{z_2} > 0$ and $0, z_1, z_2$ lies on a straight line (*i.e.* collinear) such that z_1, z_2 lies on the same side of $z = 0$.

(ii) $|z_1 - z_2| = |z_1| + |z_2| \Rightarrow \arg(z_1) - \arg(z_2) = \pi$

$\Rightarrow \frac{z_1}{z_2} < 0$ and z_1, z_2 are collinear such that $z_1 < 0 < z_2$, *i.e.*, 0 lies between z_1 and z_2.

(iii) $|z_1 - z_2| = |z_1 + z_2| \Rightarrow \arg(z_1) - \arg(z_2) = \pm\frac{\pi}{2}$, both z_1, z_2 and $\frac{z_1}{z_2}(z_2 \neq 0)$ are purely imaginary numbers.

Cube Roots of Unity and its Properties

- If $z = (1)^{1/3} \Rightarrow z^3 - 1 = 0$ then roots are $1, -\frac{1}{2}+\frac{i\sqrt{3}}{2}=\omega, -\frac{1}{2}-\frac{i\sqrt{3}}{2}=\omega^2$
- Euler form of cube roots of unity *i.e.*, Euler form of $1, \omega, \omega^2$ is $e^0, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}$
- Product of cube roots of 1 is $1 \cdot \omega \cdot \omega^2 = \omega^3 = 1$, $\bar{\omega} = \frac{1}{\omega} = \omega^2, \ \bar{\omega}^2 = \frac{1}{\omega^2} = \omega$
- $\omega^n = \omega^r$, where r is the remainder when the integer n is divided by 3. *i.e.*, $\omega^{3m+r} = \omega^r \cdot (\omega^3)^m = \omega^r$ (as $\omega^3 = 1$) ($m \in N$/positive integer)
- $1+\omega^r+\omega^{2r} = \begin{cases} 0, & \text{if } r \neq 3m, m \in N \\ 3, & \text{if } r = 3m, m \in N \end{cases}$
- $1, \omega, \omega^2$ form an equilaternal triangle inscribed in the circle of unit radius *i.e.* $|z| = 1$.
- If $a \neq 0, b \neq 0, c \neq 0$ such that $a^2 + b^2 + c^2 = a + b + c = 0$, then $a : b : c = 1 : \omega : \omega^2$.
- $1, \omega, \omega^2$ are cube roots of 1 and $-1, -\omega, -\omega^2$ are cube roots of -1. $2, 2\omega, 2\omega^2$ are cube roots of 8 and $-2, -2\omega, -2\omega^2$ are cube roots of -8.

The n^{th} roots of unity and its properties

Let $z = (1)^{1/n} = (\cos0° + i\sin0°)^{1/n}$
$= (\cos2\pi r + i\sin2\pi r)^{1/n}, r \in I$

$$\Rightarrow z = \cos\left(\frac{2r\pi}{n}\right) + i\sin\left(\frac{2r\pi}{n}\right)$$

(Using De-Moivre's theorem)

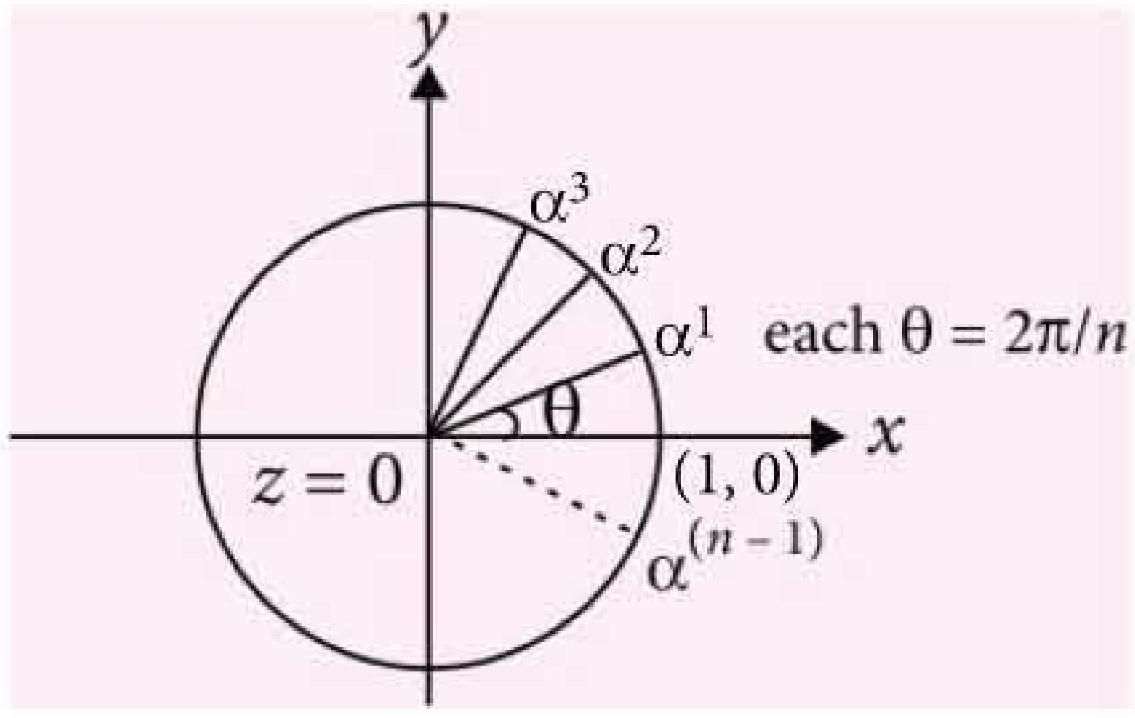
$$z = e^{\frac{i2r\pi}{n}} = \left(e^{\frac{i2\pi}{n}}\right)^r = \alpha^r,$$

where $\alpha = e^{\frac{i2\pi}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

and $r = 0, 1, 2, \ldots (n-1)$

$\therefore$ n^{th} roots of unity are $1, \alpha, \alpha^2, \ldots., \alpha^{n-1}$.

- n^{th} roots of unity, *i.e.*, $1, \alpha, \alpha^2, \ldots., \alpha^{n-1}$ form a G.P. with common ratio $e^{\frac{i2\pi}{n}}$.
- Sum of n^{th} roots of unity = 0.
- Sum of p^{th} powers of n^{th} roots of unity is zero if p is not a multiple of n, *i.e.*, $1^p + \alpha^{1p} + \alpha^{2p} + \ldots. + \alpha^{(n-1)p} = 0$



De-Moivre's Theorem

$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta}$ where n is a rational number.

Note :

(i) The value of $(\cos\theta + i\sin\theta)^{b/q}$ where $p, q \in I$, $q \neq 0$, H.C.F. $(p, q) = 1$ are given by

$$\cos\left[\frac{p}{q}(2k\pi+\theta)\right] + i\sin\left[\frac{p}{q}(2k\pi+\theta)\right],$$

where $k = 0, 1, 2, \ldots, (q-1)$ which are q distinct values and their sum is 1.

(ii) $(\sin\theta \pm i\cos\theta)^n \neq \sin n\theta \pm i\cos n\theta$

Square Root of a Complex Number and Results Related to it

Let $z = a + ib$, then

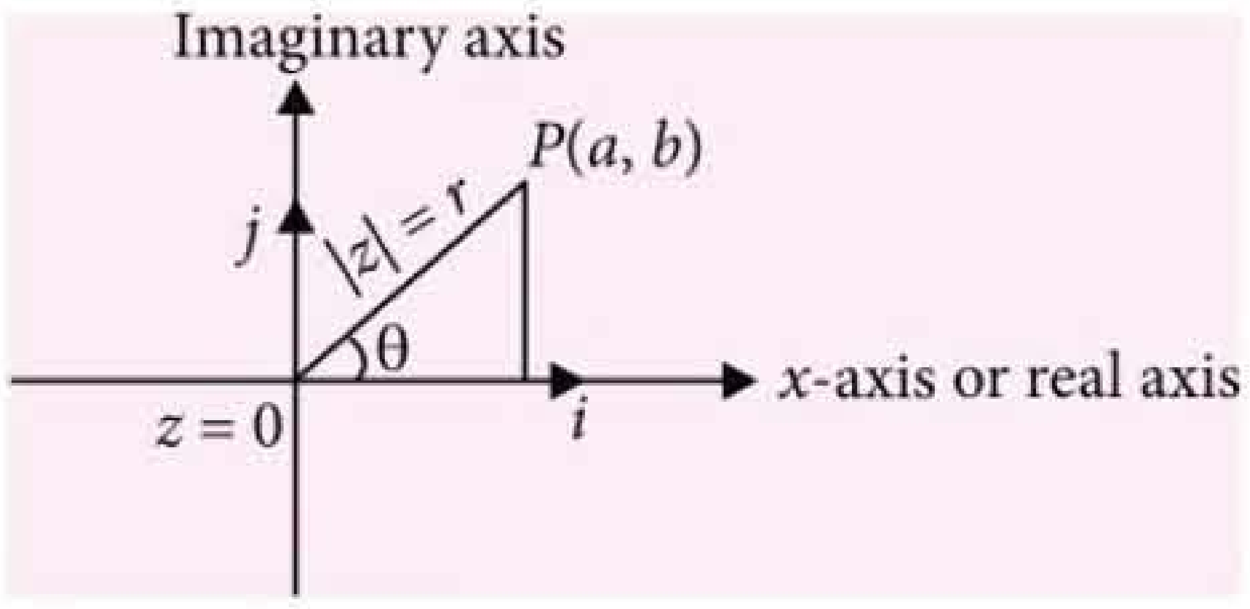
$$\sqrt{z} = \sqrt{a+ib} = \pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right], \text{ when } b > 0$$

$$= \pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right], \text{ when } b < 0$$

where $|z| = \sqrt{a^2+b^2}$

Geometry of Complex Numbers

- **Vector representation:** Every complex number $z = a + ib$ can be taken as a position vector of a point.



If $P(a, b)$ be a point in the plane corresponding to the complex number $z = a + ib$ then $\overrightarrow{OP} = a\hat{i} + b\hat{j}$ is vector representation of $z = a + ib$ and magnitude of $\overrightarrow{OP}$ *i.e.*, $|\overrightarrow{OP}|$ is called length of vector $\overrightarrow{OP}$.

$\therefore$ $|\overrightarrow{OP}| = |a\hat{i} + b\hat{j}| = \sqrt{a^2+b^2} = |a+ib| = |z| = r$

and direction vector of $\overrightarrow{OP} = \tan^{-1}(b/a) = \arg(z)$, denoted by θ.

- **Distance formula :** The distance between $P(z_1)$ and $Q\ (z_2)$ is given by

  $$|\overrightarrow{PQ}| = PQ = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

- **Section formula :** Let z_1 and z_2 represent the points $P(z)$ and $Q(z)$. Let $R(z)$ and $S(z)$ be the points which divides the join of segment PQ internally and externally in the ratio $l : m$ then affix of

  $$R(z) = \frac{lz_2 + mz_1}{l + m} \text{ and } S(z) = \frac{lz_2 - mz_1}{l - m}$$

  Corollary : If $R(z)$ is midpoint of $P(z_1)$ and $Q(z_2)$ then $R(z) = \dfrac{z_1 + z_2}{2}$.

- **Collinear points :** The points $A(z_1)$, $B(z_2)$, $C(z_3)$ will be collinear if $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$.

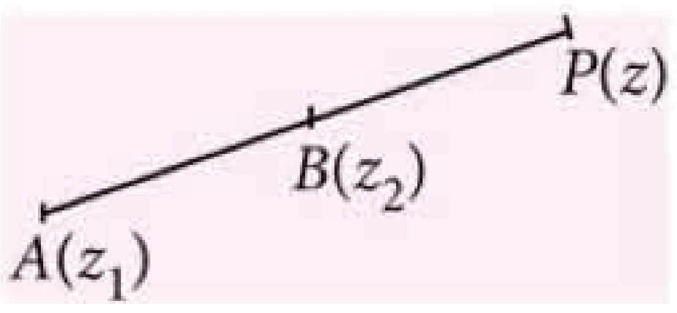
Equation of a Straight Line

Let $A(z_1)$ and $B(z_2)$ be two given points then

- Equation of straight line in non parametric form is given by

  $$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

  or $\dfrac{z - z_1}{z_2 - z_1} = \overline{\left(\dfrac{z - z_1}{z_2 - z_1}\right)}$



- **Parametric form :** $z = tz_1 + (1 - t)z_2$

  or $z = tz_2 + (1 - t)z_1$

- **General equation of straight line :** The general equation of straight line $a\bar{z} + \bar{a}z + b = 0$, where a is non-zero complex number and $b \in R$.
- The slope of such line is $-\dfrac{a}{b} = \dfrac{-\text{coefficient of } \bar{z}}{\text{coefficient of } z}$ which is called complex slope of the line.
- The complex slope of a line which makes an angle α with real axis is $e^{2i\alpha}$.
- If $A(z_1)$, $B(z_2)$ are two points in the complex plane, then complex slope of the line AB is denoted by μ and is given by $\mu = \dfrac{z_1 - z_2}{\overline{z_1 - z_2}}$ *i.e.*, $\dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.
- Equation of perpendicular bisector of a line segment joining z_1 and z_2 is given by

  $$(\bar{z}_1 - \bar{z}_2)z + (z_1 - z_2)\bar{z} = |z_1|^2 - |z_2|^2.$$

- If $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of triangle and sides opposite to these vertices are a, b and c respectively then

(i) Area of $\Delta = \dfrac{1}{4}\left\|\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}\right\|$

(ii) $G(z)$ = Centroid of triangle is point of concurrency of medians $= \dfrac{z_1 + z_2 + z_3}{3}$

(iii) Circumcentre of triangle is point of concurrency of perpendicular bisectors of the sides of the triangle.

$$S = \text{circumcentre} = \begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix} \div \begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix}$$

$$= \frac{\sum |z_1|^2 (z_2 - z_3)}{\sum \bar{z}_1 (z_2 - z_3)} = \frac{\sum z_1 \bar{z}_1 (z_2 - z_3)}{\sum \bar{z}_1 (z_2 - z_3)}$$

(iv) Orthocentre of triangle is point of concurrency of altitudes (perpendiculars) drawn from vertices to opposite sides of the triangle. H = orthocentre

$$= \left[\begin{vmatrix} z_1^2 & \bar{z}_1 & 1 \\ z_2^2 & \bar{z}_2 & 1 \\ z_3^2 & \bar{z}_3 & 1 \end{vmatrix} + \begin{vmatrix} |z_1|^2 & z_1 & 1 \\ |z_2|^2 & z_2 & 1 \\ |z_3|^2 & z_3 & 1 \end{vmatrix}\right] \div \begin{vmatrix} \bar{z}_1 & z_1 & 1 \\ \bar{z}_2 & z_2 & 1 \\ \bar{z}_3 & z_3 & 1 \end{vmatrix}$$

$$= \frac{\sum z_1^2 (\bar{z}_2 - \bar{z}_3) + \sum |z_1|^2 (z_2 - z_3)}{\sum \bar{z}_1 (z_2 - z_3)}$$

(v) Incentre of a triangle is the point of concurrency of internal bisectors of the angles of triangles.

$$I(z) = \frac{az_1 + bz_2 + cz_3}{a + b + c}$$

(vi) The triangle with vertices z_1, z_2, z_3 will be right angled isosceles triangle at vertex z_1 if $(z_2 - z_1)^2 + (z_3 - z_1)^2 = 0$ *i.e.*, $z_2^2 + z_3^2 = 2z_1(z_2 + z_3 - z_1)$

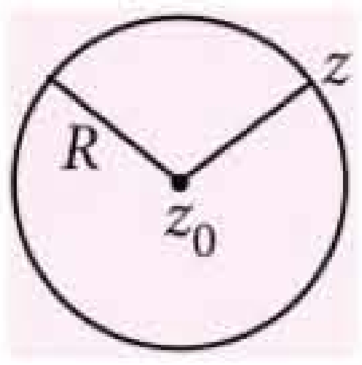
Circle

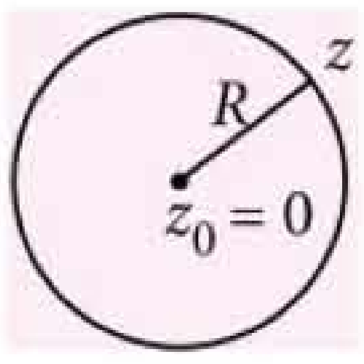
- The equation of circle whose affixes of centre is z_0 and radius R is given by $|z - z_0| = R$ ...(i)
- If the affixes of centre is $z_0 = 0$ and radius is R, then equation of circle is $|z| = R$. As $|z - z_0| = R$ from (i)

  $\Rightarrow |z - z_0|^2 = R^2$

  $\Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = R^2$

  $\Rightarrow z\bar{z} - z_0\bar{z} - z\bar{z}_0 + z_0\bar{z}_0 = R^2$



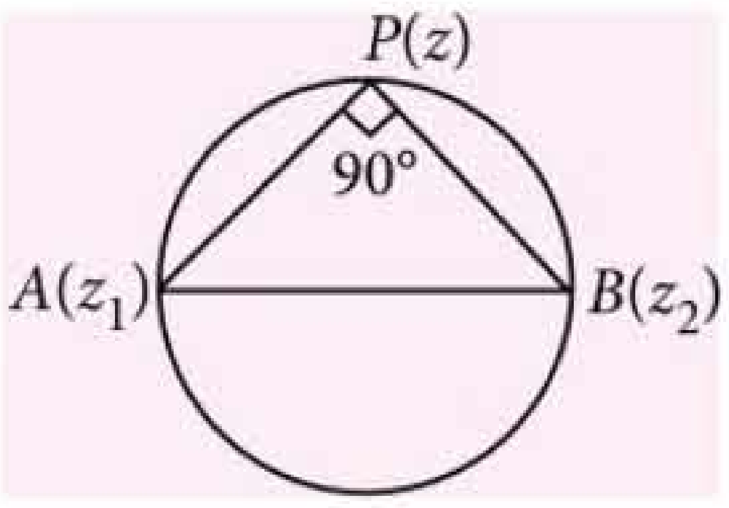


$$\Rightarrow \quad z\bar{z} + (-z_0)\bar{z} + z(-\bar{z}_0) + |z_0|^2 = R^2$$

$$\Rightarrow \quad z\bar{z} + a\bar{z} + \bar{a}z + b = 0, \qquad \ldots(*)$$

where $a = -z_0$ and $b = |z_0|^2 - R^2$.
Here, (*) is representing a circle,
whose centre is $-a$, radius $= \sqrt{|a|^2 - b} = \sqrt{a\bar{a} - b}$

- The circle $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ cuts the circle $z\bar{z} + \bar{c}z + c\bar{z} + d = 0$ ($b, d \in R$) orthogonally if and only if $\text{Re}(\bar{a}c) = \frac{1}{2}(b + d)$.

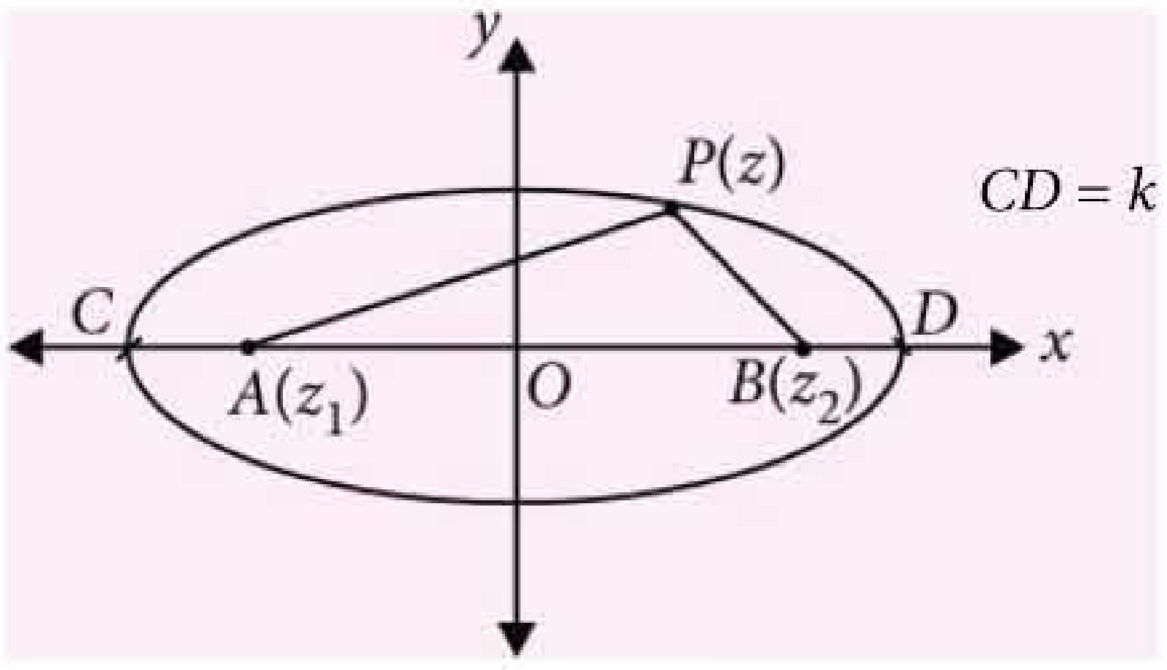


- Equation of circle whose affixes of end points of diameter are $A(z_1)$ and $B(z_2)$ is
$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$
- The equation of all circles which are orthogonal to the circles $|z - z_1| = r_1$ and $|z - z_2| = r_2$ is given by
$r_2^2 - r_1^2 = \alpha(\bar{z}_1 - \bar{z}_2) + \bar{\alpha}(z_1 - z_2) + |z_2|^2 - |z_1|^2$
where $\alpha = a + ib$ and $|z - \alpha| = r$ cuts orthogonally the circles r_1 and r_2.

Ellipse and Hyperbola

Let z be a variable point and z_1, z_2 are two fixed points and $k > 0$ is a fixed quantity then

- $|z - z_1| + |z - z_2| = k, k > |z_1 - z_2|$ represent an ellipse with foci at $A(z_1)$ and $B(z_2)$.



- $|z - z_1| - |z - z_2| = k$, $k \neq |z_1 - z_2|$, represent a hyperbola with foci at $A(z_1)$ and $B(z_2)$.
- $|z - z_1|^2 + |z - z_2|^2 = k$, represent a circle with affixes $A(z_1)$ and $B(z_2)$ as extremities of diameter if $k \geq \frac{1}{2}|z_1 - z_2|^2$. [In particular for $k = |z_1 - z_2|^2$].

Logarithm of Complex Number

Let $z = \alpha + i\beta = r \cdot e^{i(\theta + 2n\pi)}$, $n \in I$

$\therefore$ $\log z = \log(r \cdot e^{i(\theta + 2n\pi)})$
$= \log r + i(\theta + 2n\pi) = \log r + i\theta + 2in\pi$
$= \log|z| + i \arg z + 2in\pi$

If we put $n = 0$, we get principal value of log$z$.

$\therefore$ Principal value of $\log z = \log|z| + i \arg z$.

PROBLEMS

Single Correct Answer Type

1. If λ_1 and λ_2 be roots of the equation $x^2 + 2x + 2 = 0$ then least value of λ_3 for which $\left(\frac{\lambda_1}{\lambda_2}\right)^{\lambda_3} = 1$ is

(a) 2
(b) 3
(c) 4
(d) none of these

2. The product of the roots of the equation $|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 7 = 0$ ($x > 0$) is of the form 4^n then $n =$

(a) 4
(b) 3
(c) 2
(d) 1

3. The sum of all real values of x satisfying the equation $3^{(x-2)(x^2+4x-45)} = 1$ is

(a) 10
(b) 45
(c) –18
(d) –2

4. If $\alpha, \beta \in C$ are distinct roots of the equation $x^2 - x + 1 = 0$ then $\alpha^{2013} + \beta^{2019}$ is

(a) 0
(b) 1
(c) < 0
(d) > 0

5. If $f(x)$ is a quadratic polynomial such that $f(1) + f(2) = 0$ and -2 is a root of $f(x) = 0$ then other root of $f(x) = 0$ is

(a) 11/7
(b) –11/7
(c) 7/11
(d) –7/11

6. The sum of the roots of the equation $4^{36x-2} + 4^{12x+2} = 4^{24x+1} + 1$ is

(a) 1/12
(b) 1/6
(c) 1/4
(d) 1/2

7. Let $z(\alpha, \beta) = \cos\alpha + e^{i}\beta \sin\alpha$, ($\alpha, \beta \in R$, $i = \sqrt{-1}$) then the exhaustive set of values of modulus of $z(\theta, 2\theta)$ as θ varies, is

(a) $\left[\frac{1}{2}, \frac{3}{2}\right]$
(b) $[-\sqrt{2}, \sqrt{2}]$
(c) $\left[-\sqrt{\frac{3}{2}}, \frac{-1}{\sqrt{2}}\right]$
(d) $\left[\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right]$

8. The sum of infinite terms of the series

$$\sin\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3} + \frac{1}{3}\sin\frac{3\pi}{3} + \frac{1}{4}\sin\frac{4\pi}{3} + \ldots.\infty \text{ equals}$$

(a) $\pi/12$
(b) $\pi/3$
(c) $\pi/15$
(d) 0

9. If the complex number z for which $\arg\left(\frac{3z - 9 - 3i}{4z - 20 + 8i}\right) = \frac{\pi}{4}$ and $|z - 1 + 2i| = 2$ are $\left(a - \frac{2}{\sqrt{10}}\right) + i\left(b + \frac{6}{\sqrt{10}}\right)$ and $\left(a + \frac{2}{\sqrt{10}}\right) + i\left(b - \frac{6}{\sqrt{10}}\right)$ respectively then set of ordered pair (a, b) is

(a) (1, 2)
(b) (–1, –2)
(c) (1, –2)
(d) (–2, 1)

10. Let α, β are roots of the equation $x^2 - 8ax + 3 - 3a + 16a^2 = 0$. The value of '$a$' for which roots are exceed by 4 is

(a) $> \frac{19}{16}$ (b) $< \frac{19}{16}$

(c) <1 (d) none of these

More Than One Correct Answer Type

11. If two roots of the equation $(a - 2)(x^4 + x^2 + 1) + (a + 2)(x^2 + x + 1)^2 = 0$ are real and unequal, then set of all values of 'a 'is

(a) $(-1, 0)$ (b) $(-1, 0) \cup (0, 1)$

(c) $(-1, 1) - \{0\}$ (d) $(0, 1)$

12. The inequality $\left|\frac{x^2+mx+1}{x^2+x+1}\right| < 3$ is satisfied for all real values of x, then

(a) $-1 < m < 5$

(b) Number of integral values of m except 0 is four.

(c) The equation $x^2 - (m^2 - 4m - 5)\, x - 1 = 0$ has exactly one real root in $(0, 1)$.

(d) $-5 < m < 1$

13. The roots of the equation,
$(x^2 + 1)^2 = x(4x^2 + 5x + 4)$ are both real and imaginary of the form $A \pm \sqrt{B}$ and $C \pm iD$ respectively, then

(a) $D = \frac{\sqrt{3}}{2}$ (b) $A = \frac{5}{2}$

(c) $B = \frac{21}{4}$ (d) $C = -\frac{1}{2}$

14. If $a, b, c \in R$ such that their sum is zero, then the quadratic equation $3ax^2 + 2bx + c = 0$ has

(a) at least one root in $[-2, 1]$

(b) at least one root in $[0, 2]$

(c) at least one root in $[0, 1]$

(d) at least one root in $[-\infty, \infty]$

15. Let $f(x) = Ax^2 + Bx + C$, $A, B, C \in R$ and $A \neq 0$, let $f(x)$ is positive for all real values.
If $g(x) = f(x) + f'(x) + f''(x)$, then

(a) $g(x)$ is positive $\forall\, x \in R$

(b) $g(x)$ is negative $\forall\, x \in R$

(c) $g(x) = 0$ has non real complex roots

(d) $g(x) = 0$ has real roots

16. It is given that the complex numbers which satisfy the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ form a rectangle in the Guassian plane with the length of its diagonal having an integral number of units then

(a) length of diagonals is 10 units.

(b) area of rectangle is 48 sq. units.

(c) rectangle is symmetrical about both axes.

(d) if z_1, z_2, z_3, z_4 are vertices of a rectangle, then $\sum_{i=1}^{4} z_i = 0$.

17. Let $x_1 = z_1 + z_2 + z_3$, $x_2 = z_1 + \omega z_2 + \omega^2 z_3$ and $x_3 = z_1 + \omega^2 z_2 + \omega z_3$, where $1, \omega, \omega^2$ are cube roots of unity, then

(a) $z_1 = \frac{x_1 + x_2 + x_3}{3}$ (b) $z_2 = \frac{x_1 + \omega^2 x_2 + \omega x_3}{3}$

(c) $z_3 = \frac{x_1 + \omega x_2 + \omega^2 x_3}{3}$

(d) $z_1\, z_2\, z_3 = x_1{}^2 + x_2{}^2 + x_3{}^2 - (x_1x_2 + x_2x_3 + x_1x_3)$

Comprehension Type

Paragraph for Q. No. 18 to 20

Consider quadratic equation $ax^2 + bx + c = 0$ ....(1) where $a \neq 0$, $a, b, c, \in R$. If α, β are roots of equation of (1) then sum of the roots equal to negative of the coefficient of x divided by coefficient of x^2 and product of roots of (1) is the ratio of constant term to the coefficient of x^2. The equation (1) can be rewrite as $ax^2 + bx + c = a(x - \alpha)(x - \beta)$. Again, if $a_i (i = 1, 2, 3, ..., n)$ are in A.P. then difference between the consecutive terms is same which may be zero or a non zero quantity, if b_i $(i = 1, 2, 3, ..., n)$ are in G.P. then $\frac{b_2}{b_1} = \frac{b_3}{b_2} = ... = \frac{b_n}{b_{n-1}}$ is a fixed number may be equal to 1, less than 1 or greater than 1 and if c_i $(i = 1, 2, ..., n)$ are in H.P. then $\frac{1}{c_2} - \frac{1}{c_1} = \frac{1}{c_3} - \frac{1}{c_2} = ... = \frac{1}{c_n} - \frac{1}{c_{n-1}}$ is also a fixed quantity.On the basis of above information, answer the following questions.

18. Let p and q be the roots of the equation $x^2 - 4x + \alpha_1 = 0$ and r and s be the roots of the equation $x^2 - 36x + \alpha_2 = 0$ such that $p < q < r < s$ are in A.P., then values of α_1 and α_2 respectively are

(a) $-12, 308$ (b) $12, 308$

(c) $12, -6$ (d) $12, -308$

19. Let l_1, l_2 be the roots of equation $x^2 - 2x + A = 0$ and l_3, l_4 be the roots of equation $x^2 - 8x + B = 0$ such that l_1, l_2, l_3, l_4 are in G.P., then integral values of A and B respectively are

(a) $8, 128$ (b) $-8, -128$

(c) $-8, 128$ (d) $8, -128$

20. If β_1, β_3 are roots of equation $Ax^2 - 6x + 1 = 0$ and β_2, β_4 are roots of $Bx^2 - 10x + 1 = 0$ such that $\beta_1, \beta_2, \beta_3, \beta_4$ are in H.P., then values of A and B respectively are

(a) $5, -21$ (b) $-5, -21$ (c) $5, 21$ (d) $-5, 21$

Paragraph for Q. No. 21 to 23

$\sqrt{-1}=i$ called imaginary unit such that sum of consecutive four powers (*i.e.*, $i^n + i^{n+1} + i^{n+2} + i^{n+3}$) is zero $\forall\ n \in I$. Also $x = (1)^{1/3} \Rightarrow x^3 - 1 = 0$
$\Rightarrow (x-1)(x^2+x+1) = 0$
$\Rightarrow\ (x-1)(x-\omega)(x-\omega^2) = 0$, *i.e.*, $x = 1, \omega, \omega^2$ are cube roots of unity and imaginary roots of $x = (1)^{1/3}$ satisfies $\omega^n + \omega^{n+1} + \omega^{n+2} = 0\ \forall\ n \in I$.
Also $1\cdot\omega\cdot\omega^2 = \omega^3 = 1$.

21. If $(1+\omega)(1+\omega^2)(1+\omega^3)(1+\omega^4) \ldots (1+\omega^{6057}) = 2^{\lambda}$ then the value of λ equals

(a) 2019 (b) 2018 (c) 2017 (d) 2016

22. If $\omega(\neq 1)$ is a cube root of unity then the value of $\frac{1}{\sqrt{3}}\,|1 + 2\omega + 3\omega^2 + \ldots + (3n-1)\omega^{3n-2} + 3n\omega^{3n-1}|$ $\forall\ n \in N$, can not exceed by

(a) $n+2$ (b) $n+1$ (c) n (d) 2

23. If $z = \frac{\sqrt{3}+i}{2}$, then value of $(z^{113} + i^{163})^{235} =$

(a) z (b) z^2 (c) z^3 (d) z^4

Matrix Match Type

24. Match the following :

	Column-I		Column-II
(A)	If n^{th} roots of unity are $1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$, then $(2-\omega)(2-\omega^2) \ldots (2-\omega^{n-1})$ equals	P.	${}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots$
(B)	If z_i $(i = 1, 2, \ldots, n)$ lies on $\lvert z\rvert = 2$, then value of $\left\lvert\sum_{i=1}^{n} z_i\right\rvert - 4\left\lvert\sum_{i=1}^{n}\frac{1}{z_i}\right\rvert$ equals	Q.	2^{n-1}
(C)	If $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ be the n, n^{th} roots of unity, then the value of $\sum_{i=0}^{n-1}\frac{\alpha_i}{3-\alpha_i}$ is equal to	R.	${}^nC_{n-1} + {}^nC_{n-2} + \ldots + {}^nC_0$
		S.	$\frac{n}{3^n-1}$
		T.	$\sqrt{\left({}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n\right)} - 1$
		U.	$\sqrt{{}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + \ldots + {}^{2n+1}C_{2n+1}} - 1$

Numerical Answer Type

25. Let S be the greatest integral value of the sum of roots of equation $x^{2019} + \left(\frac{1}{3} - x\right)^{2019} = 0$ taken two at a time then sum of the digits of S equals

26. Find the sum of all divisors of the sum of all integral values of α for which one root of the equation $(\alpha - 5)x^2 - 2\alpha x + \alpha - 4 = 0$ is smaller than 1 and other is greater than 2.

27. Let $f(x) = x^2 - ax + b$, a is an odd positive integer and the roots of the equation $f(x) = 0$ are two non-equal prime numbers. If it is known that $a + b = 17$, then $f(4)\sum_{r=1}^{4} f(r)$ must be equal to

28. If $x = a + b\omega + c\omega^2$, $y = a\omega + b\omega^2 + c$ and $z = a\omega^2 + b + c\omega$, then the value of $\frac{x^2}{yz} + \frac{y^2}{xz} + \frac{z^2}{xy}$ is equal to

29. If $\sum_{a=1}^{20}(3a+4)\left[\sum_{b=1}^{6}\left\{\sin\left(\frac{2b\pi}{7}\right) - i\cos\left(\frac{2b\pi}{7}\right)\right\}\right]^{4a} = \alpha$ where $i = \sqrt{-1}$, then value of α equals

30. If $z_m = \cos\left(\frac{m\pi}{10}\right) + i\sin\left(\frac{m\pi}{10}\right)$, then $\prod_{m=1}^{4} z_m$ is equal to

SOLUTIONS

1. (c) : $\because$ λ_1, λ_2 are roots of the equation $x^2 + 2x + 2 = 0$

$\Rightarrow$ $(x+1)^2 + 1 = 0$ $\Rightarrow$ $(x+1)^2 = -1 = i^2$

$\Rightarrow$ $x + 1 = \pm i$ $\Rightarrow$ $x = -1 \pm i = -1 + i, -1 - i$

$$\therefore \left(\frac{\lambda_1}{\lambda_2}\right)^{\lambda_3} = 1 \Rightarrow \left(\frac{-1+i}{-(1+i)}\right)^{\lambda_3} = 1$$

[Considering $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$]

$$\Rightarrow \left(\frac{1-i}{1+i}\right)^{\lambda_3} = 1 \Rightarrow (-i)^{\lambda_3} = 1$$

$\therefore$ Least value of λ_3 is 4.

2. (b) : We have $|\sqrt{x}-3| + \sqrt{x}(\sqrt{x}-6) + 7 = 0$

$\Rightarrow$ $|\sqrt{x}-3| + x - 6\sqrt{x} + 7 = 0$

$\Rightarrow$ $|\sqrt{x}-3| + x - 6\sqrt{x} + 9 + 7 - 9 = 0$

$\Rightarrow$ $|\sqrt{x}-3| + (\sqrt{x}-3)^2 - 2 = 0$

$\Rightarrow$ $|\sqrt{x}-3|^2 + |\sqrt{x}-3| - 2 = 0$

$\Rightarrow$ $\lambda^2 + \lambda - 2 = 0$, where $\lambda = |\sqrt{x}-3|$

$\Rightarrow$ $(\lambda + 2)(\lambda - 1) = 0 \Rightarrow \lambda = 1, \lambda = -2$

$\Rightarrow$ $|\sqrt{x}-3| = 1$, $|\sqrt{x}-3| = -2$, which is not possible

$\Rightarrow$ $\sqrt{x} - 3 = \pm 1$ $\Rightarrow$ $\sqrt{x} = 4$ and $\sqrt{x} = 2$

$\Rightarrow$ $x = 16,\ x = 4$

$\therefore$ Product of roots $= 16 \times 4 = 4^3 = 4^n$

$\therefore$ $n = 3$

3. (d) : $3^{(x-2)(x^2+4x-45)} = 1 = 3^0$

$\Rightarrow$ $(x-2)(x^2 + 4x - 45) = 0$

$\Rightarrow$ $(x-2)(x+9)(x-5) = 0 \Rightarrow x = 2, 5, -9 \in R$

$\therefore$ Required sum of all real values $= 2 + 5 - 9 = -2$

4. (c) : $\because x^2 - x + 1 = 0$

$\Rightarrow$ $x^2 + (\omega + \omega^2)x + 1 = 0$ (using $\omega + \omega^2 = -1$)

$\Rightarrow$ $x(x + \omega) + \omega^2(x + \omega) = 0$

$\Rightarrow$ $(x + \omega)(x + \omega^2) = 0 \Rightarrow x = -\omega, -\omega^2$

$\therefore$ $(-\omega)^{2013} + (-\omega^2)^{2019} = -[1 + 1] = -2 < 0$

or $(-\omega^2)^{2013} + (-\omega)^{2019} = -[1 + 1] = -2 < 0$

$\therefore$ $\alpha^{2013} + \beta^{2019} = \alpha^{2019} + \beta^{2013} = -2 < 0$

5. (a) : Let $f(x) = ax^2 + bx + c$ $\therefore$ $f(1) = a + b + c$, $f(2) = 4a + 2b + c$

$\therefore$ $f(1) + f(2) = 0$ $\Rightarrow$ $5a + 3b + 2c = 0$ ...(i)

Since -2 is a root of $f(x) = 0$.

$\therefore$ $a(-2)^2 + b(-2) + c = 0 \Rightarrow 4a - 2b + c = 0$ ...(ii)

From (i) and (ii), we get

$$3a - 7b = 0 \Rightarrow \frac{b}{a} = \frac{3}{7}$$

If other root of $f(x) = 0$ is α, then sum of roots

$$= \alpha + (-2) = -\frac{b}{a} \Rightarrow \alpha - 2 = \frac{-3}{7} \Rightarrow \alpha = \frac{-3}{7} + 2 = \frac{11}{7}$$

6. (b) : Let $4^{12x} = t$

$\therefore$ $4^{36x-2} + 4^{12x+2} = 4^{24x+1} + 1$

$$\Rightarrow \frac{t^3}{16} + 16t = 4t^2 + 1$$

$\Rightarrow$ $t^3 - 64t^2 + 256t - 16 = 0$ ...(i)

Let (i) has roots t_1, t_2, t_3, then

$$t_1\, t_2\, t_3 = \frac{-\text{constant term}}{\text{coefficient of } t^3} = 16$$

$$\Rightarrow 4^{12x_1} \cdot 4^{12x_2} \cdot 4^{12x_3} = 4^2$$

$$\Rightarrow 4^{12(x_1+x_2+x_3)} = 4^2 \Rightarrow x_1 + x_2 + x_3 = \frac{1}{6}$$

$$\therefore \text{ Sum of roots} = \frac{1}{6}$$

7. (d) : $\because$ $z(\alpha, \beta) = \cos\alpha + e^{i\beta}\sin\alpha$

$\therefore$ $z(\theta, 2\theta) = \cos\theta + e^{i2\theta}\sin\theta = \cos\theta + (\cos 2\theta + i\sin 2\theta)\sin\theta$
$= (\cos\theta + \cos 2\theta \sin\theta) + i\sin 2\theta\sin\theta$

$$\Rightarrow |z(\theta,\ 2\theta)| = \sqrt{(\cos\theta + \cos 2\theta \sin\theta)^2 + (\sin 2\theta \sin\theta)^2}$$

$$= \sqrt{\cos^2\theta + \sin^2\theta(\cos^2 2\theta + \sin^2 2\theta) + \frac{\sin 4\theta}{2}} = \sqrt{1 + \frac{\sin 4\theta}{2}}$$

Now, $-1 \le \sin 4\theta \le 1$ $\Rightarrow$ $-\frac{1}{2} \le \frac{\sin 4\theta}{2} \le \frac{1}{2}$

$$\Rightarrow \frac{1}{2} \le 1 + \frac{\sin 4\theta}{2} \le \frac{3}{2} \Rightarrow \frac{1}{\sqrt{2}} \le \sqrt{1 + \frac{\sin 4\theta}{2}} \le \sqrt{\frac{3}{2}}$$

$$\Rightarrow |z(\theta,\ 2\theta)| \in \left[\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right]$$

8. (b) : Let $S = \sin\frac{\pi}{3} + \frac{1}{2}\sin\frac{2\pi}{3} + \frac{1}{3}\sin\frac{3\pi}{3} + \ldots\ \infty$

and $C = \cos\frac{\pi}{3} + \frac{1}{2}\cos\frac{2\pi}{3} + \frac{1}{3}\cos\frac{3\pi}{3} + \ldots\ \infty$

$$\therefore C + iS = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) + \frac{1}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) + \frac{1}{3}\left(\cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}\right) + \ldots\ \infty$$

$$= e^{i\frac{\pi}{3}} + \frac{1}{2}e^{i\frac{2\pi}{3}} + \frac{1}{3}e^{i\frac{3\pi}{3}} + \ldots\infty$$

$$= y + \frac{y^2}{2} + \frac{y^3}{3} + \ldots\infty \left(\text{where } y = e^{i\frac{\pi}{3}}\right)$$

$= -\log(1 - y)$

$$=-\log\left(1-e^{i\frac{\pi}{3}}\right)=-\log\left(1-\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$$

$$=-\log\left[\left(1-\frac{1}{2}\right)-i\frac{\sqrt{3}}{2}\right]\quad=-\log\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$

$$=-\log z\quad\left(\text{where } z=\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)$$

$= -[\log|z| + i\arg(z)]$ (using $\log z = \log|z| + i\arg(z)$)

$$=-\left[\log 1+i\tan^{-1}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)\right]=0-i\tan^{-1}\left(\frac{-\sqrt{3}/2}{1/2}\right)$$

$$=-i\tan^{-1}(-\sqrt{3})=i\tan^{-1}(\sqrt{3})\ =i\frac{\pi}{3}$$

$$\therefore\ C+iS=i\frac{\pi}{3}\quad\Rightarrow S=\frac{\pi}{3}$$

9. (c) : Given, $\arg\left(\dfrac{3z-9-3i}{4z-20+8i}\right)=\dfrac{\pi}{4}$

$$\Rightarrow\quad \arg\left[\frac{3}{4}\left(\frac{z-3-i}{z-5+2i}\right)\right]=\frac{\pi}{4}$$

$$\Rightarrow\quad \arg\left(\frac{3}{4}\right)+\arg\left(\frac{z-3-i}{z-5+2i}\right)=\frac{\pi}{4}$$

$$\Rightarrow\quad \arg\left(\frac{z-3-i}{z-5+2i}\right)=\frac{\pi}{4}$$

(∵ Argument of a real number = 0)

$$\Rightarrow\quad \arg\left[\frac{(x-3)+i(y-1)}{(x-5)+i(y+2)}\right]=\frac{\pi}{4}\qquad(\text{where } z=x+iy)$$

$$\Rightarrow\quad \arg\left[\frac{[(x-3)+i(y-1)][(x-5)-i(y+2)]}{(x-5)^2+(y+2)^2}\right]=\frac{\pi}{4}$$

$$\Rightarrow \arg\left[\frac{[(x-3)(x-5)+(y-1)(y+2)]+i[(y-1)(x-5)-(x-3)(y+2)]}{(x-5)^2+(y+2)^2}\right]=\frac{\pi}{4}$$

$\Rightarrow\ (y-1)(x-5)-(x-3)(y+2)=(x-3)(x-5)+(y-1)(y+2)$

$\Rightarrow\ x^2+y^2-5x+3y+2=0$ ...(1)

and $|z-1+2i|=2\ \Rightarrow\ (x-1)^2+(y+2)^2=4$

$\Rightarrow\ x^2+y^2-2x+4y+1=0$ ...(2)

Now, from (1) and (2) on subtracting or equating, we get $3x+y-1=0\ \Rightarrow\ y=1-3x$ ...(3)

Now, from (1) and (3), we get

$x^2+(1-3x)^2-5x+3(1-3x)+2=0$

$\Rightarrow\ 10x^2-20x+6=0\ \Rightarrow\ 5x^2-10x+3=0$

$$\Rightarrow\quad x=\frac{10\pm2\sqrt{10}}{10}$$

Now, using $y=1-3x$, we get

$$y=\frac{-20\mp6\sqrt{10}}{10}=-2\mp\frac{6}{\sqrt{10}}$$

$$\Rightarrow\quad x=1\pm\frac{2}{\sqrt{10}}\ \text{and}\ y=-2\mp\frac{6}{\sqrt{10}}$$

$$\therefore\ x+iy=\left(1\pm\frac{2}{\sqrt{10}}\right)+i\left(-2\mp\frac{6}{\sqrt{10}}\right)$$

$$=\left(a\pm\frac{2}{\sqrt{10}}\right)+i\left(b\mp\frac{6}{\sqrt{10}}\right)\ \Rightarrow\ a=1,\ b=-2$$

10. (a) : Using concept, the function $f(x)=Ax^2+Bx+C=0$ has both roots greater than the real value k if

(i) $D>0$ (ii) $\dfrac{\alpha+\beta}{2}>k$ (iii) $A\,f(k)>0$

Now (i) $D>0\ \Rightarrow 64a^2-4(1)(3-3a+16a^2)>0$

$\Rightarrow\ a>1$

(ii) $\dfrac{\alpha+\beta}{2}>4\ \Rightarrow\ \dfrac{8a}{2}>4\ \Rightarrow\ a>1$

(iii) $Af(k)>0\ \Rightarrow\ 1\cdot f(4)>0$

$\Rightarrow\ 1(16-32a+3-3a+16a^2)>0$

$\Rightarrow\ 16a^2-35a+19>0\Rightarrow(16a-19)(a-1)>0$

$\Rightarrow\ a<1$ or $a>\dfrac{19}{16}$ ∴ (i), (ii) and (iii) $\Rightarrow\ a>\dfrac{19}{16}$

11. (b, c) : We have, $(a-2)(x^4+x^2+1)+(a+2)(x^2+x+1)^2=0$

$\Rightarrow\ (a-2)(x^2+x+1)(x^2-x+1)+(a+2)(x^2+x+1)^2=0$

$\Rightarrow\ (x^2+x+1)[(a-2)(x^2-x+1)+(a+2)(x^2+x+1)]=0$

$\Rightarrow\ (x^2+x+1)=0$ or $(a-2)(x^2-x+1)+(a+2)(x^2+x+1)=0$

$\Rightarrow\ x^2+x+1=0$ or $a(x^2-x+1+x^2+x+1)-2(x^2-x+1-x^2-x-1)=0$

$\Rightarrow\ 2a(x^2+1)+2(2x)=0$

$\Rightarrow\ ax^2+2x+a=0\ \ (a\neq0)$ ....(i)

For real and unequal roots of (i), we have

$4(1-a^2)>0\ \Rightarrow\ |a|<1\ \Rightarrow\ -1<a<1$ ....(ii)

$\therefore\ a\in(-1,1)-\{0\}$ or $(-1,0)\cup(0,1)$

12. (a, b, c) : From given inequality, we have

$$-3<\frac{x^2+mx+1}{x^2+x+1}<3$$

$$\because\ x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\text{ is }\geq\frac{3}{4}$$

$\Rightarrow\ -3(x^2+x+1)<x^2+mx+1<3(x^2+x+1)$

$\Rightarrow\ 4x^2+(m+3)x+4>0$ ...(1)

and $2x^2+(3-m)x+2>0\ \forall\ x\in R$ ...(2)

In both cases $f(x) > 0$ and $a > 0$ therefore discriminant must be negative, where a is coefficient of x^2.

$\therefore \quad D_1 < 0$ and $D_2 < 0$

$\Rightarrow \quad (m+3)^2 - 64 < 0$ and $(3-m)^2 - 16 < 0$

$\Rightarrow \quad m^2 + 6m - 55 < 0$ and $m^2 - 6m - 7 < 0$

$\Rightarrow \quad (m+11)(m-5) < 0$ and $(m-7)(m+1) < 0$

$\Rightarrow \quad m \in (-11, 5)$ and $m \in (-1, 7)$

$\Rightarrow \quad m \in (-1, 5)$ (option (a) is correct)

Now, integral values of $m = 0, 1, 2, 3, 4$

(option (b) is correct)

Further let $f(x) = x^2 - (m^2 - 4m - 5)x - 1$

Then, $f(0) = -1$ and $f(1) = -(m^2 - 4m - 5)$

$\therefore \quad f(0)\,f(1) = (-1)^2(m^2 - 4m - 5)$

$= (m^2 - 4m - 5) = (m-5)(m+1) < 0 \ \forall\, m \in (-1, 5)$

$\therefore \quad f(x) = 0$ has only one real root in $(0, 1)$.

(option (c) is correct)

13. (a, b, c, d) : Given equation is

$(x^2+1)^2 = x(4x^2+5x+4)$ ...(i)

$\Rightarrow \quad x^4 - 4x^3 - 3x^2 - 4x + 1 = 0$

$$\Rightarrow \quad x^2\left(x^2 - 4x - 3 - \frac{4}{x} + \frac{1}{x^2}\right) = 0$$

$$\Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right) - 4\left(x + \frac{1}{x}\right) - 3 = 0$$

(as $x = 0$ does not satisfy (i))

$$\Rightarrow \quad t^2 - 4t - 5 = 0, \text{ where } t = x + \frac{1}{x}$$

$\Rightarrow \quad (t+1)(t-5) = 0$

$$\Rightarrow \quad \left(x + \frac{1}{x} + 1\right)\left(x + \frac{1}{x} - 5\right) = 0$$

$\Rightarrow \quad (x^2 + x + 1)(x^2 - 5x + 1) = 0$

$\Rightarrow \quad x^2 + x + 1 = 0, \ x^2 - 5x + 1 = 0$

$$\Rightarrow \quad x = \frac{-1 \pm i\sqrt{3}}{2},\ x = \frac{5 \pm \sqrt{21}}{2}$$

$$x = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = C \pm iD,\ x = \frac{5}{2} \pm \sqrt{\frac{21}{4}} = A \pm \sqrt{B}$$

$$\Rightarrow C = \frac{-1}{2}, D = \frac{\sqrt{3}}{2},\quad A = \frac{5}{2},\ B = \frac{21}{4}$$

$\therefore$ All options are correct.

14. (a, b, c, d) : Let $f(x) = ax^3 + bx^2 + cx + d = 0$, $f(x)$ is a polynomial function and so f is continous and differentiable on R.

Also, $f(0) = d$ and $f(1) = a + b + c + d = d$

$(\because \quad a + b + c = 0)$

$\Rightarrow \quad f(0) = f(1) \Rightarrow$ Rolle's theorem is applicable.

Thus there exist at least one value of $\alpha \in (0, 1)$ for which $f'(\alpha) = 0$

Here, $f'(\alpha) = 0 \Rightarrow 3a\alpha^2 + 2b\alpha + c = 0$

$\Rightarrow \quad 3ax^2 + 2bx + c = 0$ has at least one root in $[0, 1]$

Again, $[0, 1] \subseteq [-2, 1]$, $[0, 1] \subseteq [0, 2]$, $[0, 1] \subseteq (-\infty, \infty)$

$\therefore$ All options are correct.

15. (a, c) : $f(x)$ is positive for all $x \in R$

$\therefore \quad Ax^2 + Bx + C > 0$

$\Rightarrow \quad B^2 - 4AC < 0$ and $A > 0$ ...(i)

Now, $f'(x) = 2Ax + B,\ f''(x) = 2A$

$\therefore \quad g(x) = Ax^2 + Bx + C + 2Ax + B + 2A$

$= Ax^2 + (2A + B)x + (2A + B + C)$

Here, we have $A > 0$ and $D = (2A + B)^2 - 4A(2A + B + C) = (B^2 - 4AC) - 4A^2 =$ sum of nagative numbers < 0 (using (i))

$\therefore$ For $g(x)$ we have, coefficient of $x^2 > 0$ and discriminant < 0

$\Rightarrow \quad g(x) > 0 \ \forall\, x \in R \Rightarrow g(x) = 0$ has non real complex roots.

16. (a, b, c, d) : Let the complex number be $z = x + iy$, which satisfy the given equation $z\bar{z}^3 + \bar{z}z^3 = 350$

$\Rightarrow \quad z\,\bar{z}\,\bar{z}^2 + \bar{z}\,z\,z^2 = 350$

$\Rightarrow \quad z\,\bar{z}(\bar{z}^2 + z^2) = 350$

$\Rightarrow \quad (x^2 + y^2)\,[(x - iy)^2 + (x + iy)^2] = 350$

$\Rightarrow \quad 2(x^2 + y^2)\,(x^2 - y^2) = 350$

$\Rightarrow \quad (x^2 + y^2)\,(x^2 - y^2) = 175 = 25 \times 7 = 5 \times 35$

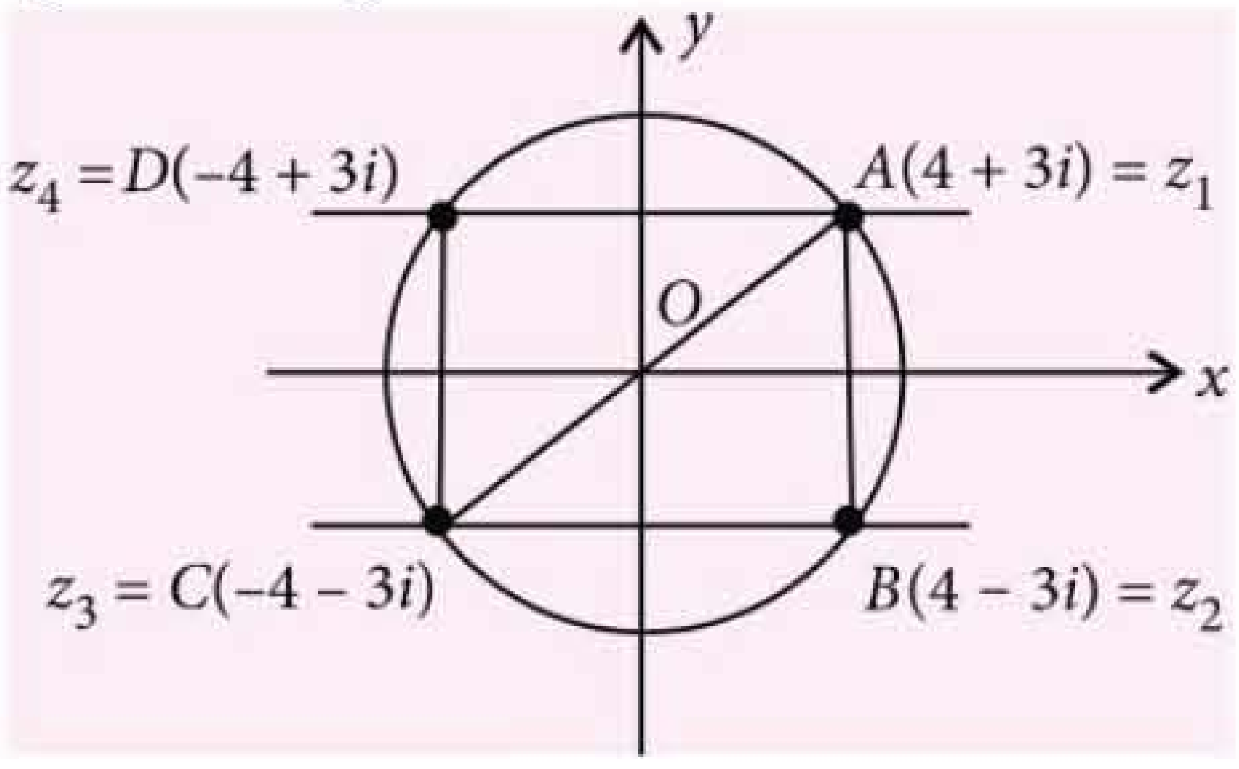
$\Rightarrow \quad x^2 + y^2 = 25$ and $x^2 - y^2 = 7$

$\Rightarrow \quad x^2 = 16,\ y^2 = 9 \quad \therefore \quad x = \pm 4,\ y = \pm 3$

[The other possibilities does not hold the conditions]

$\therefore \quad x + iy = 4 + 3i,\ 4 - 3i,\ -4 - 3i,\ -4 + 3i,$

$\therefore$ Length of diagonal $= \sqrt{64 + 36} = 10$ units



$$\text{Consider, } \sum_{i=1}^{4} z_i = (4+3i) + (4-3i) + (-4-3i) + (-4+3i) = 0$$

Area of rectangle $= 6 \times 8 = 48$ sq. units

From the figure, rectangle is symmetrical about both axes.

$\therefore$ All options are correct

17. (a, b, c) : We have, $x_1 = z_1 + z_2 + z_3$ ....(i)

$x_2 = z_1 + \omega z_2 + \omega^2 z_3$ ...(ii)

and $x_3 = z_1 + \omega^2 z_2 + \omega z_3$ ...(iii)

On adding (i), (ii) and (iii), we get

$x_1 + x_2 + x_3 = 3z_1 + z_2(1 + \omega + \omega^2) + z_3(1 + \omega + \omega^2)$

$\Rightarrow \quad x_1 + x_2 + x_3 = 3z_1 \qquad (\because\ 1 + \omega + \omega^2 = 0)$

$$\Rightarrow \quad z_1 = \frac{x_1 + x_2 + x_3}{3}$$

Again, multiplying (i), (ii) and (iii) by 1, ω^2, ω respectively and then adding, we get

$z_1(1 + \omega^2 + \omega) + z_2(1 + \omega^3 + \omega^3) + z_3(1 + \omega^4 + \omega^2)$

$= x_1 + \omega^2 x_2 + \omega x_3$

$$\Rightarrow \quad z_2 = \frac{x_1 + \omega^2 x_2 + \omega x_3}{3}$$

Similarly, $z_3 = \dfrac{x_1 + \omega x_2 + \omega^2 x_3}{3}$

and $z_2 z_3 = \dfrac{1}{9}(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - x_3x_1)$

18. (a) : As $p < q < r < s \in$ A.P.

$\therefore \quad q = p + D,\ r = p + 2D,\ s = p + 3D$ where D is common difference.

Now, as p, q are roots of $x^2 - 4x + \alpha_1 = 0$

$\therefore \quad p + q = 4$ and $pq = \alpha_1 \Rightarrow 2p + D = 4$ ...(i)

and as r, s are roots of $x^2 - 36x + \alpha_2 = 0$

$\therefore \quad r + s = 36$ and $rs = \alpha_2$

$\Rightarrow \quad 2p + 5D = 36$ ...(ii)

Solving (i) and (ii), we get $D = 8$, $p = -2$

$\therefore \quad q = p + D = 6,\ r = p + 2D = 14,\ s = p + 3D = 22$

Now, $\alpha_1 = pq = -12$ and $\alpha_2 = rs = 14 \times 22 = 308$

19. (b) : As $l_1, l_2, l_3, l_4 \in$ G.P.

$\therefore\ l_2 = l_1R,\ l_3 = l_1R^2,\ l^4 = l_1R^3$, where R is common ratio.

$\because \quad l_1, l_2$ are roots of $x^2 - 2x + A = 0$

$\Rightarrow l_1l_2 = A$ and $l_1 + l_2 = 2 \Rightarrow l_1(1 + R) = 2$ ...(1)

and $A = l_1l_2 = l_1{}^2R$ ...(*)

Again l_3, l_4 are roots of the equation $x^2 - 8x + B = 0$

$\therefore \quad l_3 + l_4 = l_1R^2(1 + R) = 8$ ...(2)

and $l_3l_4 = l_1{}^2R^5 = B$ ...(**)

On dividing (2) by (1), we get

$$\frac{l_1R^2(1+R)}{l_1(1+R)} = \frac{8}{2} \Rightarrow R^2 = 4 \Rightarrow R = \pm 2$$

Case (i) : If $R = 2$, then $l_1(1 + R) = 2 \Rightarrow l_1 = \dfrac{2}{3}$

$\therefore\ l_1, l_2, l_3, l_4$ are $\dfrac{2}{3}, \dfrac{4}{3}, \dfrac{8}{3}, \dfrac{16}{3}$

$\Rightarrow \quad l_1l_2 = \dfrac{8}{9} = A,\ l_3l_4 = \dfrac{128}{9} = B$, rejecting the value of $R = 2$ as A and B do not satisfy the condition of integral values.

Case (ii) : If $R = -2$, then $l_1(1 + R) = 2 \Rightarrow l_1 = -2$

$\therefore \quad l_1, l_2, l_3, l_4$ are $-2, 4, -8, 16$

$\therefore \quad l_1l_2 = (-2)(4) = -8 = A$, which is an integer

and $l_3l_4 = (-8)(16) = -128 = B$, which is an integer

20. (c) : $\because\ \beta_1, \beta_3$ are roots of $Ax^2 - 6x + 1 = 0$

$\therefore \quad \beta_1 + \beta_3 = \dfrac{6}{A}$ and $\beta_1\beta_3 = \dfrac{1}{A}$ ...(i)

$\Rightarrow \quad \dfrac{2\beta_1\beta_3}{\beta_1 + \beta_3} = \dfrac{1}{3}$ ...(*)

Similarly, $\beta_2 + \beta_4 = \dfrac{10}{B}$ and $\beta_2\beta_4 = \dfrac{1}{B}$ ...(ii)

$\Rightarrow \quad \dfrac{2\beta_2\beta_4}{\beta_2 + \beta_4} = \dfrac{1}{5}$ ...(**)

$\because \quad \beta_1, \beta_2, \beta_3, \beta_4 \in$ H.P.

$\Rightarrow \quad \beta_2 = \dfrac{2\beta_1\beta_3}{\beta_1 + \beta_3} = \dfrac{1}{3}$ which is a root of $Bx^2 - 10x + 1 = 0$

$$\therefore \quad B\left(\frac{1}{3}\right)^2 - 10\left(\frac{1}{3}\right) + 1 = 0 \Rightarrow B = 21$$

and $\beta_3 = \dfrac{2\beta_2\beta_4}{\beta_2 + \beta_4} = \dfrac{1}{5}$, which is a root of $Ax^2 - 6x + 1 = 0$

$$\Rightarrow \quad A\left(\frac{1}{5}\right)^2 - 6\left(\frac{1}{5}\right) + 1 = 0 \Rightarrow A = 5$$

21. (a) : Given, $(1 + \omega^1)(1 + \omega^2)(1 + \omega^3)(1 + \omega^4)$ .... $(1 + \omega^{6056})(1 + \omega^{6057}) = 2^\lambda$

Taking groups of three numbers we get 2019 groups.

$$\therefore \quad \underbrace{(1+\omega)(1+\omega^2)(1+\omega^3)}_{1^{\text{st}}\text{ group}}\underbrace{(1+\omega^4)(1+\omega^5)(1+\omega^6)}_{2^{\text{nd}}\text{ group}}$$

$$....\underbrace{(1+\omega^{6055})(1+\omega^{6056})(1+\omega^{6057})}_{(2019)^{\text{th}}\text{ group}}$$

$$= \underbrace{(-\omega^2)(-\omega)(2)}_{}\underbrace{(-\omega^2)(-\omega)(2)}_{}\ \text{... up to 2019 groups}$$

$= 2 \times 2 \times 2 \times$ .... (2019 times) $= 2^{2019} = 2^\lambda \Rightarrow \lambda = 2019$

22. (c) : Let $S = 1 + 2\omega + 3\omega^2 + ... + (3n - 1)\omega^{3n-2} + 3n\omega^{3n-1}$ ...(i)

$\Rightarrow \quad \omega S = \omega + 2\omega^2 + ... + (3n - 2)\omega^{3n-2} + (3n - 1)\,\omega^{3n-1} + 3n\omega^{3n}$ ...(ii)

Subtracting (ii) from (i), we get

$S(1 - \omega) = 1 + \omega + \omega^2 + \omega^3 + ... + \omega^{3n-1} - 3n\omega^{3n}$

$= \dfrac{\omega^{3n} - 1}{\omega - 1} - 3n\omega^{3n}$ (using sum of G.P.)

$$\therefore \quad S = \frac{-3n}{1-\omega} \Rightarrow |S| = \frac{3n}{|1-\omega|} \Rightarrow |S| = \frac{3n}{\sqrt{3}} = \sqrt{3}n$$

$$\left\{\because\ |1-\omega| = \left|1 - \left(\frac{-1}{2} \pm i\frac{\sqrt{3}}{2}\right)\right| = \left|\frac{3}{2} \mp \frac{i\sqrt{3}}{2}\right| = \sqrt{3}\right\}$$

$$\Rightarrow \quad \frac{|S|}{\sqrt{3}}=n \quad \textit{i.e., can not be greater than } n.$$

23. (a) : Given, $z=\dfrac{i+\sqrt{3}}{2}=\dfrac{i(i+\sqrt{3})}{2i}=\dfrac{1}{i}\left(\dfrac{i^2+i\sqrt{3}}{2}\right)$

$$=\frac{1}{i}\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$\therefore \quad z=\frac{\omega}{i} \quad \text{or} \quad z=-i\omega \qquad \text{....}(*)$$

Now, $z^{113}=(-i\omega)^{113}=-i^{113}\cdot\omega^{113}=-i\omega^2$

and $i^{163}=i^{(4\times 40+3)}=-i$

$\therefore \quad z^{113}+i^{163}=-i(1+\omega^2)=i\omega$

$\therefore \quad (z^{113}+i^{163})^{235}=(i\omega)^{235}=i^{235}\,\omega^{235}=-i\omega=z.$

24. A → Q, R, T, U, B → P, C → S

(A) As $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are the n, n^{th} roots of unity,

$\therefore \quad z^n-1=(z-1)(z-\omega)(z-\omega^2)\ldots(z-\omega^{n-1})$

$\Rightarrow \quad 2^n-1=(2-\omega)(2-\omega^2)\ldots(2-\omega^{n-1})$

i.e., $(2-\omega)(2-\omega^2)\ldots(2-\omega^{n-1})=2^n-1$

$$={}^nC_1+{}^nC_2+\ldots+{}^nC_n={}^nC_{n-1}+{}^nC_{n-2}+\ldots+{}^nC_0$$

Also, $2^{2n+1}=({}^{2n+1}C_0+{}^{2n+1}C_1+\ldots+{}^{2n+1}C_n)$

$$+({}^{2n+1}C_{n+1}+{}^{2n+1}C_{n+2}+\ldots+{}^{2n+1}C_{2n+1})$$

$\Rightarrow \quad 2^{2n+1}=2({}^{2n+1}C_0+{}^{2n+1}C_1+\ldots+{}^{2n+1}C_n)$

$\Rightarrow \quad (2^n)^2={}^{2n+1}C_0+{}^{2n+1}C_1+\ldots+{}^{2n+1}C_n$

$$={}^{2n+1}C_{n+1}+{}^{2n+1}C_{n+2}+\ldots+{}^{2n+1}C_{2n+1}$$

$$\Rightarrow \quad 2^n=\sqrt{{}^{2n+1}C_0+{}^{2n+1}C_1+\ldots+{}^{2n+1}C_n}$$

$$=\sqrt{{}^{2n+1}C_{n+1}+{}^{2n+1}C_{n+2}+\ldots+{}^{2n+1}C_{2n+1}}$$

$$\Rightarrow \quad 2^n-1=\sqrt{{}^{2n+1}C_0+{}^{2n+1}C_1+{}^{2n+1}C_2\ldots+{}^{2n+1}C_n}-1$$

$$=\sqrt{{}^{2n+1}C_{n+1}+{}^{2n+1}C_{n+2}+\ldots+{}^{2n+1}C_{2n+1}}-1$$

(B) Consider, $4\left|\displaystyle\sum_{i=1}^{n}\frac{1}{z_i}\right|=4\left|\frac{1}{z_1}+\frac{1}{z_2}+\ldots+\frac{1}{z_n}\right|$

$$=\left|\frac{4}{z_1}+\frac{4}{z_2}+\frac{4}{z_3}+\ldots+\frac{4}{z_n}\right|$$

$$=\left|\frac{|z_1|^2}{z_1}+\frac{|z_2|^2}{z_2}+\ldots+\frac{|z_n|^2}{z_n}\right| \qquad (\because |z|=2)$$

$$=|\bar{z}_1+\bar{z}_2+\ldots+\bar{z}_n| \qquad (\because |z^2|=z\bar{z})$$

$$=\left|\overline{z_1+z_2+z_3+\ldots+z_n}\right|$$

$$=|z_1+z_2+\ldots+z_n| \qquad (\because |z|=|\bar{z}|)$$

$$\therefore \quad |z_1+z_2+\ldots+z_n|-4\left|\frac{1}{z_1}+\frac{1}{z_2}+\ldots+\frac{1}{z_n}\right|=0$$

$$\Rightarrow \quad \left|\sum_{i=1}^{n}z_i\right|-4\left|\sum_{i=1}^{n}\frac{1}{z_i}\right|=0=(1-x)^n \text{ for } x=1$$

$\Rightarrow \quad {}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+\ldots=0$

(C) Let $A=\displaystyle\sum_{i=0}^{n-1}\left(\frac{\alpha_i}{3-\alpha_i}\right)$

$$=-\sum_{i=0}^{n-1}\frac{-\alpha_i}{3-\alpha_i}=-\sum_{i=0}^{n-1}\left[\frac{(3-\alpha_i)-3}{3-\alpha_i}\right]$$

$$=-\sum_{i=0}^{n-1}1+3\sum_{i=0}^{n-1}\frac{1}{3-\alpha_i} \qquad \text{...(1)}$$

$$\because \quad z^n-1=\prod_{i=0}^{n-1}(z-\alpha_i)$$

$$\Rightarrow \quad \log(z^n-1)=\sum_{i=0}^{n-1}\log(z-\alpha_i) \qquad \text{...}(*)$$

On differentiating w.r.t. z both sides of (*), we get

$$\frac{nz^{n-1}}{z^n-1}=\sum_{i=0}^{n-1}\frac{1}{z-\alpha_i}$$

Now, taking $z=3$, we get $\dfrac{n\cdot 3^{n-1}}{3^n-1}=\displaystyle\sum_{i=0}^{n-1}\frac{1}{3-\alpha_i}$

From equation (1), we have

$$A=3\sum_{i=0}^{n-1}\frac{1}{3-\alpha_i}-n=3\cdot\frac{n\cdot 3^{n-1}}{3^n-1}-n$$

$$=\frac{n\cdot 3^n-n(3^n-1)}{3^n-1}=\frac{n}{3^n-1} \quad \therefore \sum_{i=0}^{n-1}\frac{\alpha_i}{3-\alpha_i}=\frac{n}{3^n-1}$$

25. (28) : Given equation is $x^{2019}+\left(\dfrac{1}{3}-x\right)^{2019}=0$ ...(*)

$$\Rightarrow \quad x^{2019}-\left(x-\frac{1}{3}\right)^{2019}=0$$

$$\Rightarrow \quad x^{2019}-\left[x^{2019}-\binom{2019}{1}x^{2018}\left(\frac{1}{3}\right)\right.$$

$$\left.+\binom{2019}{2}x^{2017}\left(\frac{1}{3}\right)^2-\ldots\ldots-\left(\frac{1}{3}\right)^{2019}\right]=0$$

$$\Rightarrow \quad \binom{2019}{1}x^{2018}\frac{1}{3}-\binom{2019}{2}x^{2017}\left(\frac{1}{3}\right)^2$$

$$+\binom{2019}{3}x^{2016}\left(\frac{1}{3}\right)^3-\ldots\ldots+\left(\frac{1}{3}\right)^{2019}=0$$

$\therefore$ Sum of the roots of (*) taken two at a time

$$=\frac{\text{coefficient of } x^{2016}}{\text{coefficient of } x^{2018}}=\frac{2019\cdot 2018\cdot 2017}{3\cdot 2\cdot 1}\cdot\frac{1}{27}\cdot\frac{3}{2019}$$

$\therefore\ S=\left[\frac{1009\times 2017}{27}\right]=75376$, where $[\cdot]$ denotes greatest integer function.

$\therefore$ Sum of digits of $S = 7 + 5 + 3 + 7 + 6 = 28$

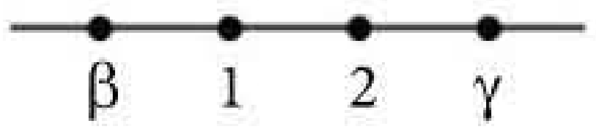
26. (390) : Let $f(x) = (\alpha - 5)x^2 - 2\alpha x + \alpha - 4\ (\alpha \neq 5)$

Let roots of $f(x) = 0$ are β, γ respectively.

As 1 and 2 lies between the roots of $f(x) = 0$, we get

$D > 0,\ (\alpha - 5)\, f(1) < 0$

and $(\alpha - 5)\, f(2) < 0$



where $f(1) = (\alpha - 5) - 2\alpha + \alpha - 4 = -9$

and $f(2) = 4(\alpha - 5) - 4\alpha + \alpha - 4 = \alpha - 24$

$D > 0 \ \Rightarrow\ 4\alpha^2 - 4(\alpha - 5)(\alpha - 4) > 0$

$\Rightarrow\ 9\alpha - 20 > 0\ \Rightarrow\ \alpha \in \left(\frac{20}{9}, \infty\right)$ ...(i)

$(\alpha - 5)\, f(1) < 0\ \Rightarrow\ (\alpha - 5)(-9) < 0$

$\Rightarrow\ \alpha - 5 > 0\ \Rightarrow\ \alpha \in (5, \infty)$ ...(ii)

$(\alpha - 5)\, f(2) < 0\ \Rightarrow\ (\alpha - 5)(\alpha - 24) < 0$

$\Rightarrow\ \alpha \in (5, 24)$ ...(iii)

From (i), (ii) and (iii), we get $\alpha \in (5, 24)$

$\Rightarrow\ \alpha = 6, 7, 8, 9, 10, 11, 12, \ldots, 23$

$\therefore$ Sum of all integral values of $\alpha = 6 + 7 + \ldots + 23$, which is an A.P. having $a = 6$, $l = 23$, $n = 18$

Let S = sum of all integral values of α

$$=\frac{n}{2}(a+l)=\frac{18}{2}(6+23)\ = 9 \times 29 = 3^2 \times 29^1$$

$\therefore$ Sum of all divisors of $S = \dfrac{3^3-1}{3-1}\times\dfrac{29^2-1}{29-1}$

$$=13\times\frac{840}{28}=390$$

27. (0) : Given, $f(x) = x^2 - ax + b$ and we are given a is an odd positive integer.

Let roots of $x^2 - ax + b = 0$ be x_1 and x_2 ($x_1 \neq x_2$ given)

$\therefore$ $x_1 + x_2 = a$ is an odd positive integer and so both x_1, x_2 cannot be odd.

Therefore one root must be 2 (as roots are prime numbers)

$\therefore\ f(2) = 0\ \Rightarrow\ 4 - 2a + b = 0\ \Rightarrow\ 2a - b = 4$ ...(i)

And we have $a + b = 17$ ...(ii)

Solving (i) and (ii), we get $a = 7$ and $b = 10$

$\therefore\ f(x) = x^2 - 7x + 10$

$\therefore\ f(4) = 16 - 28 + 10 = -2$

and $\sum\limits_{r=1}^{4} f(r)=\sum\limits_{r=1}^{4}(r^2-7r+10)$

$$=\frac{4\cdot 5\cdot 9}{6}-\frac{7\cdot 4\cdot 5}{2}+10\times 4=0$$

$$\therefore\ f(4)\sum_{r=1}^{4} f(r)=(-2)(0)=0$$

28. (3) :
$$\left.\begin{aligned} x &= a + b\omega + c\omega^2\\ y &= a\omega + b\omega^2 + c\\ z &= a\omega^2 + b + c\omega\end{aligned}\right\} \quad \text{...(i)}$$

On adding equations of (i), we get

$$x + y + z = (1 + \omega + \omega^2)(a + b + c) = 0$$

$$\therefore\ x^3 + y^3 + z^3 = 3xyz\ \Rightarrow\ \frac{x^3+y^3+z^3}{xyz}=3$$

$$\Rightarrow\ \frac{x^2}{yz}+\frac{y^2}{xz}+\frac{z^2}{xy}=3$$

29. (710) : Consider, $\sum\limits_{b=1}^{6}\left\{\sin\left(\frac{2b\pi}{7}\right)-i\cos\left(\frac{2b\pi}{7}\right)\right\}$

$$=-i\sum_{b=1}^{6}\left\{\cos\left(\frac{2b\pi}{7}\right)+i\sin\left(\frac{2b\pi}{7}\right)\right\}$$

$$=-i\sum_{b=1}^{6}e^{i\left(\frac{2b\pi}{7}\right)}=-i\left[\sum_{b=0}^{6}e^{i\left(\frac{2b\pi}{7}\right)}-1\right]$$

$= -i[0 - 1] = i$ [$\because$ sum of n^{th} roots of unity = 0]

$$\therefore\ \sum_{a=1}^{20}(3a+4)\left[\sum_{b=1}^{6}\left\{\sin\left(\frac{2b\pi}{7}\right)-i\cos\left(\frac{2b\pi}{7}\right)\right\}\right]^{4a}$$

$$=\sum_{a=1}^{20}(3a+4)(i)^{4a}\ =\sum_{a=1}^{20}(3a+4) \qquad (\because\ i^{4a} = (i^4)^a = 1)$$

$$=\frac{3\times 20\times 21}{2}+4\times 20\ = 630 + 80 = 710$$

30. (–1) : $\prod\limits_{m=1}^{4} z_m = z_1\, z_2\, z_3\, z_4$

$$=\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)\left(\cos\frac{2\pi}{10}+i\sin\frac{2\pi}{10}\right)$$
$$\left(\cos\frac{3\pi}{10}+i\sin\frac{3\pi}{10}\right)\left(\cos\frac{4\pi}{10}+i\sin\frac{4\pi}{10}\right)$$

$$=\cos\left(\frac{\pi}{10}+\frac{2\pi}{10}+\frac{3\pi}{10}+\frac{4\pi}{10}\right)+i\sin\left(\frac{\pi}{10}+\frac{2\pi}{10}+\frac{3\pi}{10}+\frac{4\pi}{10}\right)$$

[$\because$ $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ \ldots\ldots (\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$]

$$=\cos\left(\frac{10\pi}{10}\right)+i\sin\left(\frac{10\pi}{10}\right)=\cos\pi+i\sin\pi = -1$$

YOU ASK WE ANSWER

Do you have a question that you just can't get answered?
Use the vast expertise of our MTG team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough. The best questions and their solutions will be printed in this column each month.

1. Find the range of the function $f(x) = \ln(x^x + 1), x \in (0, 1)$.

(Akhilesh Yadav, Haryana)

Ans. Let $g(x) = x^x + 1, x \in (0, 1)$.
Then, $g'(x) = x^x(1 + \ln x)$.
Now, $g'(x) < 0 \ \forall\ x < 1/e$ and $g'(x) > 0 \ \forall\ x > 1/e$.
$\Rightarrow$ $g(x)$ increases in $(1/e, 1)$ and decreases in $(0, 1/b)$.

$$\text{Now, } g(0^+) = \lim_{x\to 0^+}(x^x+1) = \lim_{x\to 0^+}[e^{\left(\frac{\ln x}{1/x}\right)}+1]$$

$$= \lim_{x\to 0^+} e^{\left(\frac{1/x}{-1/x^2}\right)}+1 = 1 + 1 = 2$$

$$\text{Also, } g\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{1/e} + 1$$

$$\text{And, } g(1^-) = \lim_{x\to 1^-}(x^x+1) = 1+1 = 2$$

$$\therefore \quad \left(\frac{1}{e}\right)^{1/e} + 1 \le g(x) < 2$$

$$\Rightarrow \quad \ln((1/e)^{1/e} + 1) \le \ln\{g(x)\} < \ln 2$$

Hence, the range of $f(x)$ is $y \in \left[\ln\left((1/e)^{(1/e)}+1\right), \ln 2\right]$.

2. Consider the function $f(x) = \int_0^x (1+t^3)^{-1/2}\, dt$. If g is the inverse of f, then find the value of $\frac{g''}{g^2}$.

(Rohit Kumar, Maharashtra)

Ans. We have, $f(x) = \int_0^x (1+t^3)^{-1/2}\, dt$

$$\text{Now, } f\{g(x)\} = \int_0^{g(x)} (1+t^3)^{-1/2}\, dt$$

$$\Rightarrow \quad x = \int_0^{g(x)} (1+t^3)^{-1/2}\, dt \qquad \text{...(i)}$$

$[\because$ g is inverse of $f \Rightarrow f\{g(x)\} = x]$

Differentiating both sides of (i) w.r.t. x, we get
$1 = (1 + g^3)^{-1/2} \cdot g' \quad \Rightarrow \quad (g')^2 = 1 + g^3$
Again, differentiating both sides of w.r.t. x, we get
$2g'g'' = 3g^2g'$

Thus, $\frac{g''}{g^2} = \frac{3}{2}$.

3. Solve for x :

$$\frac{6}{5}a^{\log_a x \log_{10} a \log_a 5} - 3^{\log_{10}(x/10)} = 9^{\log_{100} x + \log_4 2}.$$

(Prachi Sharma, Delhi)

Ans. We have,

$$a^{\log_a x \log_{10} a \log_a 5} = a^{\log_a 5 \log_{10} x} = [a^{\log_a 5}]^{\log_{10} x} = 5^{\log_{10} x}$$

Also, $\log_{10}(x/10) = \log_{10} x - 1$

and $\log_{100} x + \log_4 2 = \frac{\log_{10} x}{\log_{10} 100} + \frac{\log_{10} 2}{\log_{10} 4} = \frac{\log_{10} x}{2} + \frac{1}{2}$

The given equation thus reduces to

$$\frac{6}{5}5^{\log_{10} x} - 3^{\log_{10} x - 1} = 9^{(\log_{10} x + 1)/2}$$

$$\Rightarrow \quad \frac{6}{5}5^{\log_{10} x} - 3^{\log_{10} x} \cdot 3^{-1} = 3^{(\log_{10} x + 1)}$$

$$\Rightarrow \quad \frac{6}{5}5^{\log_{10} x} - \frac{1}{3} \cdot 3^{\log_{10} x} = 3^{\log_{10} x} \cdot 3^1$$

$$\Rightarrow \quad \frac{6}{5}5^{\log_{10} x} = \left(3 + \frac{1}{3}\right)3^{\log_{10} x} \Rightarrow \frac{6}{5}5^{\log_{10} x} = \frac{10}{3}3^{\log_{10} x}$$

$$\Rightarrow \quad \frac{9}{25} = \left(\frac{3}{5}\right)^{\log_{10} x} \Rightarrow \left(\frac{3}{5}\right)^2 = \left(\frac{3}{5}\right)^{\log_{10} x}$$

$$\Rightarrow \quad \log_{10} x = 2 \Rightarrow x = 10^2 = 100$$

◆

SOLUTION SENDERS

Samurai Sudoku (July)

Subhranil Pan	
Shailaja Tekaday	(Maharashtra)
Sandeepa Dhara	(West Bengal)
Varun Chaturvedi	(Gujarat)

Mathdoku (July)

Devjit Acharjee	(West Bengal)
Tarachand Prajapat	
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Kalur Nikhil	(Hyderabad)
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Devjit Acharjee	(West Bengal)
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CONCEPT BOOSTER

Functions and Limits

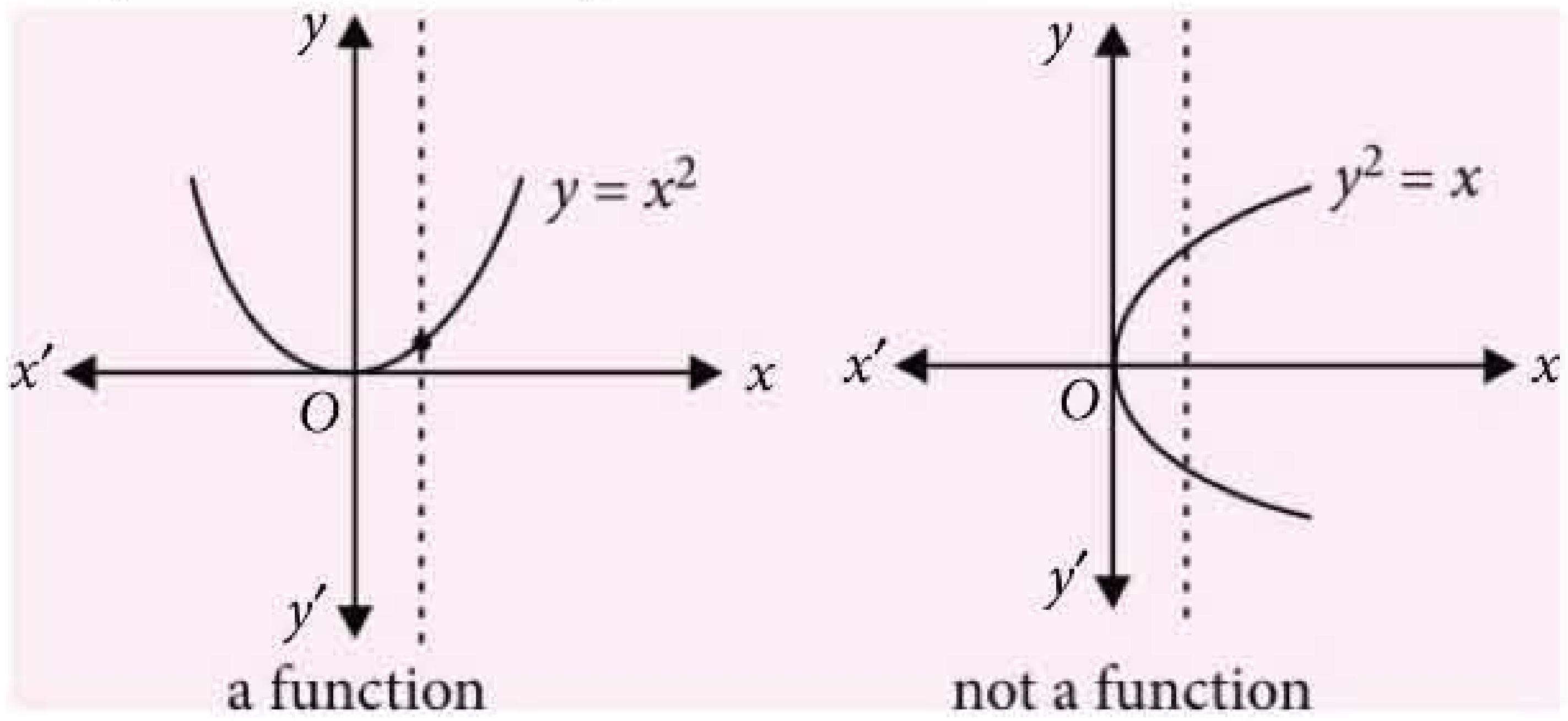
***ALOK KUMAR**

FUNCTION

Function can be easily defined with the help of the concept of mapping. Let X and Y be any two non-empty sets. "A function from X to Y is a rule or correspondence that assigns to each element of set X, one and only one element of set Y". Let the correspondence be 'f', then mathematically we write $f : X \to Y$ where $y = f(x)$, $x \in X$ and $y \in Y$ and we say that 'y' is the image of 'x' under 'f' (or x is the pre image of y). It is possible that a few elements in the set Y are present which are not the images of any element in set X.

Graphical Method

To check whether a relation between x and y is a function or not, draw a line parallel to y-axis and, if it intersect the graph at one and only one point, then the given relation represent a function.



a function not a function

Domain and Range of a Function

The set X is called the domain of the function 'f' and set Y is called the co-domain. The set of the images of all elements of X under the function 'f' is called the range of 'f' and is denoted by $f(x)$. It is obvious that range could be a subset of co-domain as we may have few elements in co-domain which are not the images of any element of the set X.

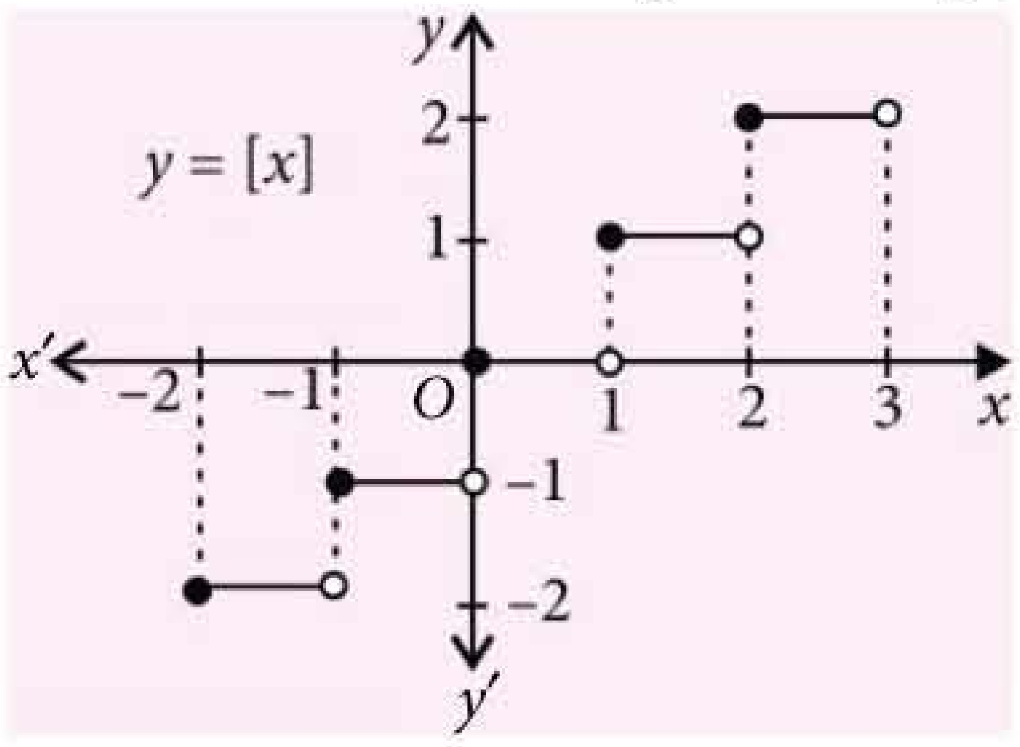
Thus range of 'f', *i.e.*, $f(x) = \{f(x) : x \in X\}$. Clearly $f(x) \subseteq Y$

GREATEST INTEGER FUNCTION

Any real number x can always be lying between two consecutive integers say P and $P+1$, *i.e.*, $P \le x < (P + 1)$. That means, there always exist an integer, say 'P' which is just less than or equal to x. This unique 'P' is called the greatest integral value of x and is symbolically denoted as $[x]$, *i.e.*, $[x]$ stands for the greatest integer that is less than or equal to x.

For example, $x = 3.54 \Rightarrow 3 < x < 4 \Rightarrow [x] = 3$,
$x = -2.95 \Rightarrow -3 < x < -2 \Rightarrow [x] = -3$

It is obvious that if x is an integer, then $[x] = x$.



Domain of $y = R$; Range of $y = I$

Note :

- $P \le x < P + 1 \Leftrightarrow [x] = P$
- $P_1 \le [x] \le P_2 \Rightarrow P_1 \le x < P_2 + 1 \Rightarrow x \in [P_1, P_2 + 1)$
- $x - 1 < [x] \le x$
- $[[x]] = [x]$
- $[n + x] = n + [x]$, where $n \in I$
- $[x] + [-x] = \begin{cases} 0, & \text{if } x \in \text{integer} \\ -1, & \text{if } x \notin \text{integer} \end{cases}$

FRACTIONAL PART FUNCTION

Fractional part of any real number is defined as the difference between the number 'x' and it's integral value '$[x]$' and is symbolically denoted as $\{x\}$.

Thus, $\{x\} = x - [x]$, *e.g.*, if $x = 5.68$, then $[x] = 5$ and $\{x\} = 0.68$.

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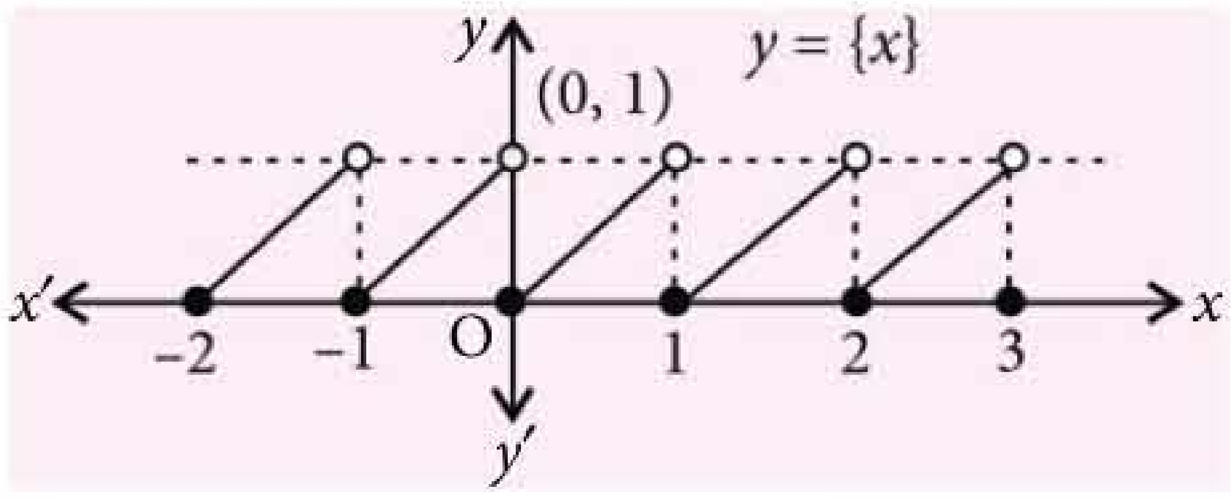
*Alok Kumar, a B Tech from IIT Kanpur and INMO 4th ranker of his time, has been training IIT and Olympiad aspirants for close to two decades now. His students have bagged AIR 1 in IIT JEE and also won medals for the country at IMO. He has also taught at Maths Olympiad programme at Cornell University, USA and UT, Dallas. He has been regularly proposing problems in international Mathematics journals.

If x is an integer, then $x = [x] \Rightarrow \{x\} = 0 \Rightarrow \{[x]\} = 0$
If $x \in [0, 1)$, then $[x] = 0 \Rightarrow \{x\} = x$
If $x \in [1, 2)$, then $[x] = 1 \Rightarrow \{x\} = x - 1$



Domain of $y = R$; Range of $y = [0,1)$

Note :

- $0 \le \{x\} < 1$
- $[\{x\}] = 0, \{[x]\} = 0$
- $x - 1 < [x] \le x, 0 \le \{x\} < 1$
- $\{x\} + \{-x\} = \begin{cases} 0, \text{ if } x \in \text{ integer} \\ 1, \text{ if } x \notin \text{integer} \end{cases}$

TYPES OF FUNCTIONS

One-One or Injective Function

A function $f : X \to Y$ is said to be one-one or injective if each element in the domain of a function has a distinct image in the co-domain. *E.g.*, $f : R \to R$ defined by $f(x) = 2x$ is one-one.

Many-One Function

A function $f : X \to Y$ is said to be many one if there are atleast two elements in the domain whose images are same. *E.g.*, $f : R \to R$ defined by $y = x^2$ is many-one.

Methods to Determine One-One and Many-One Functions

- If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for every x_1, x_2 in the domain, then 'f' is one-one else many-one.
- If the function is entirely increasing or decreasing in the domain, then 'f' is one-one else many-one.
- If we draw a line parallel to the x-axis which intersects the graph of $y = f(x)$ at one and only one point, then $f(x)$ is one-one and if the line parallel to the x-axis cuts the graph at more than one points, then $f(x)$ is a many-one function.
- Any continuous function $f(x)$, which has atleast one local maxima or local minima is many-one.
- All even functions are many one.
- All polynomials of even degree defined on R have atleast one local maximum or minimum and hence are many one on the domain R. Polynomials of odd degree can be one-one or many-one.

Onto Function or Surjective Function

A function $f : X \to Y$ is said to be a onto function or surjective function if and only if each element of Y is the image of some element of X *i.e.*, if and only if for every $y \in Y$ there exists some $x \in X$ such that $y = f(x)$. Thus 'f' is onto if $f(x) = Y$ *i.e.*, range = co-domain of function.

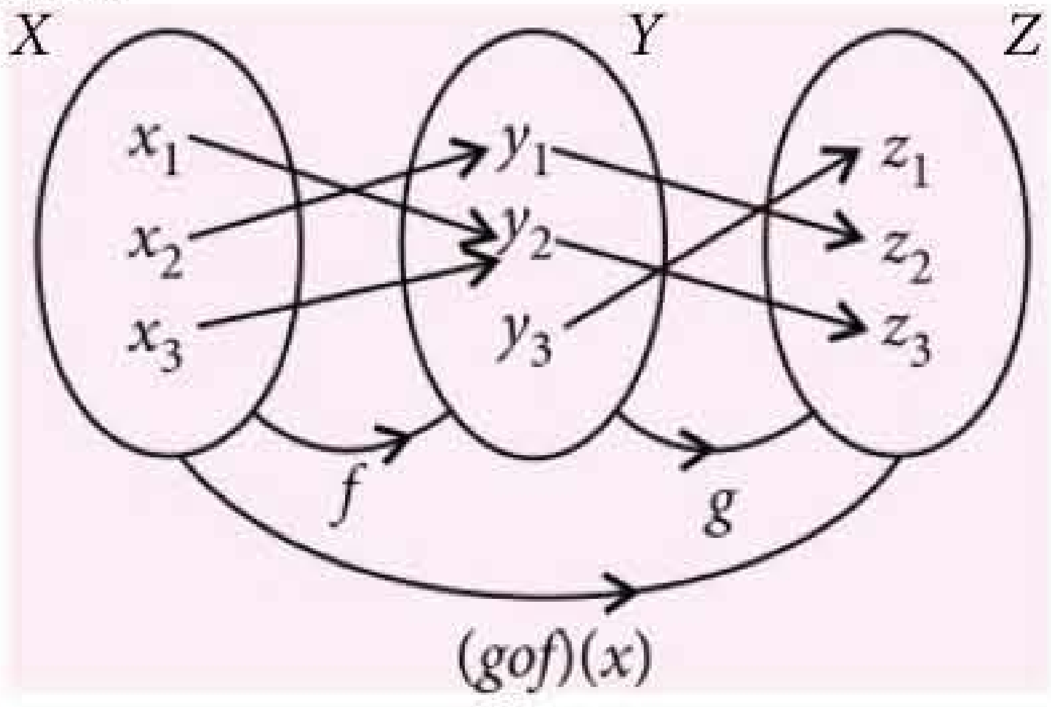
Into Function

A function $f : X \to Y$ is said to be an into function if there exists atleast one element in the co-domain Y which is not an image of any element in the domain X.
Example : The map $f : R \to R$ given by $f(x) = x^2$ is an into map.

One-One Onto or Bijective Function : A function $f : X \to Y$ is said to be one-one onto or bijective function if and only if distinct elements of X have distinct images in Y, each element of Y has atleast one pre-image in X.

COMPOSITE FUNCTION

If $f : X \to Y$ and $g : Y \to Z$ then we define the composite function $(gof) : X \to Z$ by $(gof)(x) \equiv g\{f(x)\}$. To obtain $(gof)(x)$, we first take the f-image of an element $x \in X$ so that $f(x) \in Y$, which is the domain of $g(x)$. Then take g-image of $f(x)$, *i.e.* $g(f(x))$ which would be an element of Z.



Here, $(g(f(x_1)) = g(y_2) = z_3$.
Range of $(gof) = \{z_2, z_3\}$, but range of $g = \{z_1, z_2, z_3\}$

Note :

- Clearly domain $(gof) = \{x : x \in \text{domain of } (f), f(x) \in \text{domain of } (g)\}$
- Similarly we can define, $(fog)x = f(g(x))$ and domain $(fog) = \{x : x \in \text{domain of } (g), g(x) \in \text{domain of } (f)\}$. In general $fog \ne gof$.

INVERSE FUNCTION

If $f : X \to Y$ be a function defined by $y = f(x)$ such that f is both one-one and onto, then there exists a unique function $g : Y \to X$ such that for each $y \in Y$, $g(y) = x$ if and only if $y = f(x)$. The function g so defined is called the inverse of f and denoted by f^{-1}.

Note :

- The condition for existence of inverse of a function is that the function must be one-one and onto.
- Whenever an inverse function is defined, the range of the original function becomes the domain of the inverse function and domain of the original function becomes the range of the inverse function.

- Note that $fof^{-1}(x) = f^{-1}of\,(x) = x$ always and roots of the equation $f(x) = f^{-1}(x)$ would always lie on the line $y = x$.
- f and f^{-1} are symmetric about the line $y = x$.

EVEN AND ODD FUNCTIONS

If $f : X \to Y$ is a real valued function such that for all $x \in D \Rightarrow -x \in D$ (where D = domain of f) and if $f(-x) = f(x)$ for every $x \in D$, then f is said to be an even function and if $f(-x) = -f(x)$, then f is said to be an odd function.

Important Points

- Even functions are symmetric about the y-axis (*i.e.*, if (x, y) lies on the curve, then $(-x, y)$ also lies on the curve).
- Odd functions are symmetric about the origin and it is placed either in the first and third quadrant or in the second and fourth quadrant (*i.e.*, if (x, y) lies on the curve, then $(-x, -y)$ also lies on the curve).
- $f(x) = 0$ is the only function which is both even and odd.
- If $f(x)$ is an odd function, then $f(x)$ is differentiable on R.
- To express a given function $f(x)$ as the sum of an even and odd function, we write $f(x) = \left(\dfrac{f(x)+f(-x)}{2}\right) + \left(\dfrac{f(x)-f(-x)}{2}\right)$, where $\left(\dfrac{f(x)+f(-x)}{2}\right)$ is an even function and $\left(\dfrac{f(x)-f(-x)}{2}\right)$ is an odd function.
- If $x = 0 \in$ domain of f, then for odd function $f(x)$, $f(0) = 0$ *i.e.*, if for a function, $f(0) \neq 0$, then that function can not be odd.

PERIODIC FUNCTION

A function $f : X \to Y$ is said to be a periodic function provided there exists a positive real number T such that $f(x + T) = f(x)$, for all $x \in X$. The least of all such positive numbers T is called the principal period or fundamental period or simply period of f.

Some Important Points

- To check the periodicity of a function put $f(T + x) = f(x)$ and solve this equation to find the positive values of T independent of x. If positive values of T independent of x are obtained, then $f(x)$ is a periodic function and the least positive value of T is the period of the function $f(x)$. If no positive value of T independent of x is obtained, then $f(x)$ is non-periodic function.
- A constant function is periodic but does not have a well-defined period.
- If $f(x)$ is periodic with period p, then $f(ax + b)$ where $a, b \in R$ $(a \neq 0)$ is also periodic with period $p/|a|$.
- If $f(x)$ is periodic with period p, then $af(x) + b$ where $a, b \in R$ $(a \neq 0)$ is also periodic with period p.
- Let $f(x)$ has period $p = m/n$ ($m, n \in N$ and co-prime) and $g(x)$ has period $q = r/s$ ($r, s \in N$ and co-prime) and let t be the LCM of p and q *i.e.*, $t = \dfrac{\text{LCM of } (m, r)}{\text{HCF of } (n, s)}$, then t shall be the period of $f + g$ provided there does not exist a positive number k $(< t)$ for which $f(k + x) + g(k + x) = f(x) + g(x)$, else k will be the period. The same rule is applicable for any other algebraic combination of $f(x)$ and $g(x)$.
- LCM of p and q always exist if p/q is a rational quantity. If p/q is irrational, then algebraic combination of f and g is non-periodic.
- $\sin^n x$, $\cos^n x$, $\operatorname{cosec}^n x$ and $\sec^n x$ have period 2π, if n is odd and π, if n is even.
- $\tan^n x$ and $\cot^n x$ have period π, whether n is odd or even.
- If g is periodic, then fog will always be a periodic function. Period of fog may or may not be the period of g.
- If f is periodic and g is strictly monotonic (other than linear), then fog is non-periodic.

LIMIT OF A FUNCTION

Let $y = f(x)$ be a function of x. If at $x = a$, $f(x)$ takes indeterminate form, then we consider the values of the function which are very near to 'a'. If these values tend to a definite unique number as x tends to 'a', then the unique number so obtained is called the limit of $f(x)$ at $x = a$ and we write it as $\lim\limits_{x\to a} f(x)$.

- Meaning of '$x \to a$': Let x be a variable and a be the constant. If x assumes values nearer and nearer to 'a' then we say 'x tends to a' and we write '$x \to a$'. It should be noted that as $x \to a$, we have $x \neq a$.

Left Hand Limit (L.H.L.) and Right Hand Limit (R.H.L.) : Consider the values of the function at the points which are very near to a on the left of a. If these values tend to a definite unique number as x tends to a, then the unique number so obtained is called left-hand limit of $f(x)$ at $x = a$ and symbolically we write it as $f(a^-) = \lim\limits_{x\to a^-} f(x) = \lim\limits_{h\to 0} f(a-h)$

Similarly, we can define right-hand limit of $f(x)$ at $x = a$ which is expressed as

$$f(a^+) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h).$$

Existence of Limit

$\lim\limits_{x \to a} f(x)$ exists, when $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$,

i.e., L.H.L. = R.H.L.

Indeterminate Forms : If a function $f(x)$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = a$, then we say that $f(x)$ is indeterminate or meaningless at $x = a$. Other indeterminate forms are $\infty - \infty$, $\infty \times \infty$, $0 \times \infty$, 1^∞, 0^0, ∞^0.

It is not necessary that if the value of a function at some point exists, then its limit at that point must exist.

Sandwich Theorem : If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that, $f(x) \le g(x) \le h(x)$ $\forall$ $x \in$ neighbourhood of $x = a$ and $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} h(x) = l\text{(say)}$, then $\lim\limits_{x \to a} g(x) = l.$ This theorem is normally applied when the $\lim\limits_{x \to a} g(x)$ can't be obtained by using conventional methods as function $f(x)$ and $h(x)$ can be easily found.

Fundamental Theorems on Limits : If $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$ (l and m are real numbers), then

- $\lim\limits_{x \to a} (f(x) + g(x)) = l + m$ (Sum rule)
- $\lim\limits_{x \to a} (f(x) - g(x)) = l - m$ (Difference rule)
- $\lim\limits_{x \to a} (f(x) \cdot g(x)) = l \cdot m$ (Product rule)
- $\lim\limits_{x \to a} kf(x) = k \cdot l$ (Constant multiple rule)
- $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}, m \ne 0$ (Quotient rule)
- If $\lim\limits_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim\limits_{x \to a} \dfrac{1}{f(x)} = 0$
- $\lim\limits_{x \to a} \log\{f(x)\} = \log\{\lim\limits_{x \to a} f(x)\} = \log l$
- If $f(x) \le g(x)$ for all x, then $\lim\limits_{x \to a} f(x) \le \lim\limits_{x \to a} g(x)$
- $\lim\limits_{x \to a} [f(x)]^{g(x)} = \{\lim\limits_{x \to a} f(x)\}^{\lim\limits_{x \to a} g(x)} = l^m$
- If p and q are integers, then $\lim\limits_{x \to a} (f(x))^{p/q} = l^{p/q}$, provided $(l)^{p/q}$ is a real number.

Some Important Expansions

- $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + ...$
- $a^x = 1 + \dfrac{x \log a}{1!} + \dfrac{(x \log a)^2}{2!} + ...$
- $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ...$
- $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + ..., |x| < 1$
- $\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - .....$, where $|x| < 1$
- $(1+x)^{1/x} = e^{\frac{1}{x}\log(1+x)} = e^{\left(1 - \frac{x}{2} + \frac{x^2}{3} .....\right)}$
  $= e\left(1 - \dfrac{x}{2} + \dfrac{11}{24}x^2 - ...\right)$
- $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - ....$
- $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + ...$
- $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + ...$
- $\sin^{-1} x = x + 1^2 . \dfrac{x^3}{3!} + 3^2 \cdot 1^2 \cdot \dfrac{x^5}{5!} + ...$
- $\cos^{-1} x = (\pi / 2) - \sin^{-1} x$
- $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + ...$

METHODS OF EVALUATION OF LIMITS

- **Direct Substitution Method :** If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.
- **Factorisation Method :** In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.
- **Rationalisation Method :** Rationalisation is followed when we have fractional powers $\left(\text{like } \frac{1}{2}, \frac{1}{3} \text{ etc.,}\right)$ on expressions in numerator or denominator or in both. After rationalisation the terms are factorised which on cancellation gives the result.

Method of Evaluating limits when $x \to \infty$: In this case expression should be expressed as a function $1/x$

and then after removing indeterminate form, (if it is there) replace $1/x$ by 0.

- Write down the expression in the form of rational function, *i.e.*, $\frac{f(x)}{g(x)}$, if it is not so.
- If k is the highest power of x in numerator and denominator both, then divide each term of numerator and denominator by x^k.
- Use the result $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$, where $n > 0$.

An Important Result : If m, n are positive integers and a_0, $b_0 \neq 0$ are non-zero real numbers, then

$$\lim_{x\to\infty} \frac{a_0x^m + a_1x^{m-1} + ... + a_m}{b_0x^n + b_1x^{n-1} + ... + b_n} = \begin{cases} \frac{a_0}{b_0}, \text{ if } m = n \\ 0, \text{ if } m < n \\ \infty, \text{if } m > n \end{cases}$$

PROBLEMS

Subjective Type

1. Find the domain of $f(x) = \frac{x^2+1}{x^2-4}$.

2. Find the domain of $f(x) = \left(\frac{x}{1-|x|}\right)^{1/2000}$.

3. Find the domain of $\log_{(x+3)}(x^2 + 2x + 1)$.

4. Find the domain of $\cos^{-1}\frac{6-3x}{4} + \operatorname{cosec}^{-1}\frac{x-1}{2}$.

5. If $f(x) = \frac{x^2+2x+3}{x}$, then find the range of $f(x)$.

6. Find the range of $f(x) = \frac{x^2}{x^2+1}$.

7. Find the range of $f(x) = 2x^3 - 9x^2 + 12x + 1$, $x \in [-1, 3]$.

8. Solve for x, $\log_{1/4} x \geq 1$.

9. If $f: R \to R$ where $f(x) = \frac{x+1}{x^2+x+1}$, find whether $f(x)$ is one-one or many-one.

10. Let $f: R \to [2, \infty)$ defined by $f(x) = x^2 + 2x + b$ is an onto function, then find the value of b.

11. The function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, find $f^{-1}(x)$.

12. Which of the following functions is (are) even, odd or neither?

(i) $f(x) = \frac{|x^2-1|(\cos x)}{1+\log(e^x+e^x)}$ **(ii)** $f(x) = \log\left(\frac{1-x}{1+x}\right)$

(iii) $f(x) = \log(x + \sqrt{1+x^2})$ **(iv)** $f(x) = \sin x - \cos x$

13. Find the period of the function $\sin 4x + \tan 2x$.

Single Correct Answer Type

14. The value of b and c for which the identity $f(x + 1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$, are

(a) $b = 2, c = 1$ (b) $b = 4, c = -1$
(c) $b = -1, c = 4$ (d) $b = -1, c = 1$

15. If $\phi(x) = a^x$, then $\{\phi(p)\}^3$ is equal to

(a) $\phi(3p)$ (b) $3\phi(p)$ (c) $6\phi(p)$ (d) $2\phi(p)$

16. If $f(x) = \frac{x}{x-1}$, then $\frac{f(a)}{f(a+1)} =$

(a) $f(-a)$ (b) $f(1/a)$ (c) $f(a^2)$ (d) $f\left(\frac{-a}{a-1}\right)$

17. If $f(x) = 4x^3 + 3x^2 + 3x + 4$, then $x^3 f\left(\frac{1}{x}\right)$ is

(a) $f(-x)$ (b) $\frac{1}{f(x)}$
(c) $\left(f\left(\frac{1}{x}\right)\right)^2$ (d) $f(x)$

18. Let $f(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1}{3}, & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$, then f is

(a) a rational function (b) a trigonometric function
(c) a step function (d) an exponential function

19. If $f(x) = \frac{x^2-1}{x^2+1}$, for every real numbers. Then the minimum value of f

(a) does not exist because f is bounded
(b) is not attained even though f is bounded
(c) is equal to 1 (d) is equal to -1

20. If $y = f(x) = \frac{ax+b}{cx-a}$, then x is equal to

(a) $1/f(x)$ (b) $1/f(y)$ (c) $y f(x)$ (d) $f(y)$

21. If $f: R \to R$, then $f(x) = |x|$ is

(a) one-one but not onto (b) onto but not one-one
(c) one-one and onto (d) none of these

22. Let X and Y be subsets of R, the set of all real numbers. The function $f: X \to Y$ defined by $f(x) = x^2$ for $x \in X$ is one-one but not onto if (Here R^+ is the set of all positive real numbers)

(a) $X = Y = R^+$ (b) $X = R, Y = R^+$
(c) $X = R^+, Y = R$ (d) $X = Y = R$

23. The function $f : R \to R$ defined by $f(x) = (x - 1)(x - 2)(x - 3)$ is
(a) one-one but not onto
(b) onto but not one-one
(c) both one-one and onto
(d) neither one-one nor onto

24. The function $f : R \to R$ defined by $f(x) = e^x$ is
(a) onto (b) many-one
(c) one-one and into (d) many-one and onto

25. Domain and range of $f(x)=\dfrac{|x-3|}{x-3}$ are respectively
(a) $R, \{-1, 1\}$ (b) $R - \{3\}, \{1, -1\}$
(c) R^+, R (d) none of these

26. A function f from the set of natural numbers to integers defined by $f(n)=\begin{cases}\dfrac{n-1}{2}, & \text{when } n \text{ is odd}\\ -\dfrac{n}{2}, & \text{when } n \text{ is even}\end{cases}$, is
(a) one-one but not onto
(b) onto but not one-one
(c) one-one and onto both
(d) neither one-one nor onto

27. The domain of $\sin^{-1}\left[\log_3\left(\dfrac{x}{3}\right)\right]$ is
(a) $[1, 9]$ (b) $[-1, 9]$ (c) $[-9, 1]$ (d) $[-9, -1]$

28. Domain of the function $\log|x^2 - 9|$ is
(a) R (b) $R - [-3, 3]$
(c) $R - \{-3, 3\}$ (d) none of these

29. If the domain of function $f(x) = x^2 - 6x + 7$ is $(-\infty, \infty)$, then the range of function is
(a) $(-\infty, \infty)$ (b) $[-2, \infty)$ (c) $(-2, 3)$ (d) $(-\infty, -2)$

30. Domain of the function $f(x)=\dfrac{x-3}{(x-1)\sqrt{x^2-4}}$ is
(a) $(1, 2)$ (b) $(-\infty, -2) \cup (2, \infty)$
(c) $(-\infty, -2) \cup (1, \infty)$ (d) $(-\infty, \infty) - \{1, \pm 2\}$

31. Domain of the function $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$ is
(a) $(-1, 1)$ (b) $(-1, 1) - \{0\}$
(c) $[-1, 1]$ (d) $[-1, 1] - \{0\}$

32. The largest possible set of real numbers which can be the domain of $f(x)=\sqrt{1-\dfrac{1}{x}}$ is
(a) $(0, 1) \cup (0, \infty)$ (b) $(-1, 0) \cup (1, \infty)$
(c) $(-\infty, -1) \cup (0, \infty)$ (d) $(-\infty, 0) \cup (1, \infty)$

33. Domain of the function $f(x)=\dfrac{x^2-3x+2}{x^2+x-6}$ is
(a) $\{x : x \in R, x \neq 3\}$ (b) $\{x : x \in R, x \neq 2\}$
(c) $\{x : x \in R\}$ (d) $\{x : x \in R, x \neq 2, x \neq -3\}$

34. Domain of $f(x) = (x^2 - 1)^{-1/2}$ is
(a) $(-\infty, -1) \cup (1, \infty)$ (b) $(-\infty, -1] \cup (1, \infty)$
(c) $(-\infty, -1] \cup [1, \infty)$ (d) none of these

35. The range of $f(x) = \cos x - \sin x$ is
(a) $(-1, 1)$ (b) $[-1, 1)$
(c) $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ (d) $[-\sqrt{2}, \sqrt{2}]$

36. Range of the function $f(x) = \sin^2(x^4) + \cos^2(x^4)$ is
(a) $(-\infty, \infty)$ (b) $\{1\}$ (c) $(-1, 1)$ (d) $(0, 1)$

37. Range of the function $f(x)=\dfrac{x^2+x+2}{x^2+x+1}$; $x \in R$ is
(a) $(1, \infty)$ (b) $(1, 11/7)$
(c) $(1, 7/3]$ (d) $(1, 7/5)$

38. Let the function f be defined by $f(x)=\dfrac{2x+1}{1-3x}$, then $f^{-1}(x)$ is
(a) $\dfrac{x-1}{3x+2}$ (b) $\dfrac{3x+2}{x-1}$ (c) $\dfrac{x+1}{3x-2}$ (d) $\dfrac{2x+1}{1-3x}$

39. If $f(x) = e^x$ and $g(x) = \log x$, then $fog(1)$ will be
(a) e (b) $\log_e e$ (c) 0 (d) $2e$

40. If $f(x) = [x]$ and $g(x) = |x|$ then
$(gof)\left(-\dfrac{5}{3}\right)-(fog)\left(-\dfrac{5}{3}\right)=$
(a) 1 (b) -1 (c) 2 (d) 4

41. Let f and g be functions defined by $f(x)=\dfrac{x}{x+1}$, $g(x)=\dfrac{x}{1-x}$, then $(fog)(x)$ is
(a) $\dfrac{1}{x}$ (b) $\dfrac{1}{x-1}$ (c) $x - 1$ (d) x

42. Suppose that $g(x)=1+\sqrt{x}$ and $f(g(x)) = 3+2\sqrt{x}+x$, then $f(x)$ is
(a) $1 + 2x^2$ (b) $2 + x^2$ (c) $1 + x$ (d) $2 + x$

43. The composite mapping fog of the map $f : R \to R$, $f(x) = \sin x$, $g : R \to R$, $g(x) = x^2$ is
(a) $\sin x + x^2$ (b) $(\sin x)^2$
(c) $\sin x^2$ (d) $\dfrac{\sin x}{x^2}$

44. If $f(x) = \sin^2 x$ and the composite function $g(f(x)) = |\sin x|$, then the function $g(x)$ is equal to

(a) $\sqrt{x-1}$ (b) $\sqrt{x}$ (c) $\sqrt{x+1}$ (d) $-\sqrt{x}$

45. If equation of the curve remain unchanged by replacing x and y from y and x respectively, then the curve is

(a) symmetric along x-axis
(b) symmetric along y-axis
(c) symmetric along the line $y = -x$
(d) symmetric along the line $y = x$

46. If $f(x)=\dfrac{\alpha x}{x+1}, x \neq -1$, then for what value of α is $f(f(x)) = x$?

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 1 (d) -1

47. Function $f(x) = x - [x]$ where $[x]$ shows a greatest integer function. This function is

(a) a periodic function whose period is 2
(b) a periodic function whose period is 1/2
(c) a periodic function whose period is 1
(d) not a periodic function

48. The domain of function $f(x) = \log_e(x - [x])$ is

(a) R (b) $R - Z$ (c) $(0, \infty)$ (d) Z

Assertion & Reason Type

Directions : *In the following questions, Statement-1 is followed by Statement-2. Mark the correct choice as :*

(a) Statement-1 is true, Statement-2 is true, Statement-2 is a correct explanation for Statement-1.
(b) Statement-1 is true, Statement–2 is true, Statement-2 is not a correct explanation for Statement-1
(c) Statement-1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

49. Statement-1: $A = \{x : x$ is a prime number, $x < 30\}$, then number of distinct rational numbers whose numerator and denominator belong to A is 93.

Statement-2: $\dfrac{p}{q} \in Q \ \forall\ q \neq 0$ and $p, q \in I$.

50. Statement-1: Reflection of function $y = \log_e x$ in the straight line $x + y = 0$ is $y = -e^{-x}$.

Statement-2: The image of any point (α, β) in the line $x + y = 0$ is $(-\beta, -\alpha)$.

51. Let $f(x) = 2 + \cos x, x \in R$.

Statement-1: For each $t \in R\ \exists$ a point c in $[t, t + \pi]$ such that $f'(c) = 0$.

Statement-2: $f(t) = f(t + 2\pi)\ \forall\ t \in R$.

52. Let $f(x) = \sqrt{\cos^{-1}\sqrt{1-x^2} - \sin^{-1} x}$

Statement-1 : Range of $f(x)$ is $[0, \sqrt{\pi}]$.

Statement-2 : $f(x)$ is an increasing function in its domain.

53. Statement-1 : $f(x)=\begin{cases} g(x)+g(-x), & x>0 \\ a, & x=0 \\ g(x)-g(-x), & x<0 \end{cases}$

is odd function iff $a = 0$.

Statement-2 : For an odd function $y = f(x)$ if $x = 0$ is in domain, then $f(0)$ must be equal to 0.

54. Let $[x]$ denote the integral part of x.

Statement-1: Let $f(x)=\begin{cases} \dfrac{\sin[x]}{[x]}, & \text{if } [x]\neq 0 \\ 0, & \text{if } [x]=0 \end{cases}$, then $\lim\limits_{x\to 0} f(x)$ does not exist.

Statement-2 : $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)=1.$

55. Statement-1: If a and b are positive and $[x]$ denotes greatest integer $\leq x$, then $\lim\limits_{x\to 0^+}\dfrac{x}{a}\left[\dfrac{b}{x}\right]=\dfrac{b}{a}$.

Statement-2 : $\lim\limits_{x\to\infty}\dfrac{\{x\}}{x}=0$, where $\{x\}$ denotes fractional part of x.

Comprehension Type

Paragraph for Q. No. 56 to 58

Let $f(x) = \min\{x - [x], -x - [-x]\}, -2 \leq x \leq 2;$ $g(x) = |2 - |x - 2||, -2 \leq x \leq 2;$ $h(x)=\dfrac{|\sin x|}{\sin x}, -2 \leq x \leq 2,$ $x \neq 0$, where $[x]$ denotes the greatest integer function $\leq x$.

56. The number of solutions of the equation $x^2 + (f(x))^2 = 1$ is $\{-1 \leq x \leq 1\}$

(a) 0 (b) 2 (c) 4 (d) 6

57. The sum of all the roots of the equation $g(x) - h(x) = 0$ is $\{-2 \leq x \leq 2\}$

(a) positive (b) negative
(c) zero (d) None of these

58. The value of $\int\limits_{-2}^{2} f(x)\,dx =$

(a) 0 (b) 2 (c) 1 (d) 8

Paragraph for Q. No. 59 to 61

Let $f(x)$ be a real valued and differentiable function on R such that $f(x + y) = \dfrac{f(x)+f(y)}{1-f(x)f(y)}$.

59. $f(0)$ equals

(a) 1 (b) 0

(c) –1 (d) None of these

60. $f(x)$ is

(a) odd function (b) even function

(c) odd and even function simultaneously

(d) neither even nor odd.

61. $\dfrac{f'(x)}{f'(0)}$ equals

(a) $1 - f^2(x)$ (b) $2 + f^2(x)$

(c) $2 - f^2(x)$ (d) $1 + f^2(x)$

Matrix Match Type

62. For any $0 < a \in R$ let $f_a(x) = \dfrac{a^x}{a^x + \sqrt{a}}$ for all $x \in R$, then match the following :

Column-I		Column-II	
A.	$\sum\limits_{k=1}^{1997} f_9\left(\dfrac{k}{1998}\right) =$	p.	998.5
B.	$\sum\limits_{k=1}^{1997} f_4\left(\dfrac{k}{1998}\right) =$	q.	1004.5
C.	$\sum\limits_{k=1}^{2009} f_{16}\left(\dfrac{k}{2010}\right) =$	r.	993
D.	$\sum\limits_{k=1}^{2008} f_{25}\left(\dfrac{k}{2009}\right) =$	s.	1004

63. Match the following :

Column-I		Column-II	
A.	If $L = \lim\limits_{x \to -1} \dfrac{\sqrt[3]{7-x} - 2}{x+1}$, then $12L =$	p.	–2
B.	If $L = \lim\limits_{x \to \pi/4} \dfrac{\tan^3 x - \tan x}{\cos\left(x + \dfrac{\pi}{4}\right)}$, then $-(L/4) =$	q.	2
C.	If $L = \lim\limits_{x \to 1} \dfrac{(2x-3)(\sqrt{x}-1)}{2x^2 + x - 3}$, then $20L =$	r.	1
D.	If $L = \lim\limits_{x \to \infty} \dfrac{\log x^n - [x]}{[x]}$, where $n \in N$ ($[x]$ denotes greatest integer function), then $-2L =$	s.	–1

64. Match the following :

Column-I		Column-II	
A.	$f(x) = x \operatorname{sgn}(x - 1)$	p.	$\lim\limits_{x \to 1} f(x)$ doesn't exist
B.	$f(x) = \dfrac{\sin(\sin(\tan(x^2/2)))}{\log \cos 3x}$	q.	$\lim\limits_{x \to 0} f(x)$ doesn't exist
C.	$f(x) = \dfrac{\sqrt[3]{1 + \tan^{-1} 3x} - \sqrt[3]{1 - \sin^{-1} 3x}}{\sqrt{1 - \sin^{-1} 2x} - \sqrt{1 + \tan^{-1} 2x}}$	r.	$\lim\limits_{x \to 0} f(x) = -\dfrac{1}{9}$
D.	$f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$	s.	$\lim\limits_{x \to 0} f(x) = -1$

SOLUTIONS

1. Denominator should be non-zero for any function.

$\therefore \quad x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$

Hence, domain is $R - \{-2, 2\}$.

2. Expression under even root (*i.e.*, square root, fourth root, sixth root etc.) should be non-negative.

$f(x)$ is defined when $\dfrac{x}{1 - |x|} \geq 0$ ...(1)

Case 1: When $x \geq 0$, $f(x) = \left(\dfrac{x}{1-x}\right)^{1/2000}$

For domain, $\dfrac{x}{1-x} \geq 0 \Rightarrow x \in [0, 1)$ ...(2)

Case 2: When $x < 0$, $f(x) = \left(\dfrac{x}{1+x}\right)^{1/2000}$

For domain, $\dfrac{x}{1+x} \geq 0 \Rightarrow x \in (-\infty, -1) \cup [0, \infty)$

Since, $x < 0$, we get $x \in (-\infty, -1)$ ...(3)

From (2) and (3), domain is $(-\infty, -1) \cup [0, 1)$.

3. $\log_{g(x)} f(x)$ is defined for values of x at which $f(x) > 0$ and $g(x) > 0$ and $g(x) \neq 1$.

$\therefore \quad x^2 + 2x + 1 > 0$ and $x + 3 > 0$ and $x + 3 \neq 1$

$\Rightarrow \quad (x + 1)^2 > 0$ and $x > -3$ and $x \neq -2$

$\Rightarrow \quad x \neq -1$, $x > -3$ and $x \neq -2$

Hence, the domain is $(-3, \infty) - \{-1, -2\}$.

4. $\cos^{-1} x$ is defined for $-1 \leq x \leq 1$

$\operatorname{cosec}^{-1} x$ is defined for $x \leq -1$ or $x \geq 1$

Let $f(x) = \cos^{-1} \dfrac{6-3x}{4}$ which is defined when

$-1 \leq \dfrac{6-3x}{4} \leq 1 \Rightarrow \dfrac{2}{3} \leq x \leq \dfrac{10}{3}$ ...(1)

Let $g(x) = \text{cosec}^{-1}\dfrac{x-1}{2}$ which is defined when

$\dfrac{x-1}{2} \le -1$ or $\dfrac{x-1}{2} \ge 1 \Rightarrow x \in (-\infty, -1] \cup [3, \infty)$ ...(2)

From (1) and (2), domain is $\left[3, \dfrac{10}{3}\right]$.

5. If $f(x)$ is in the form of $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions of second degree we use the concept of quadratic equation.

Let $\dfrac{x^2+2x+3}{x} = y \quad \Rightarrow \quad x^2 + (2 - y)x + 3 = 0$

Since $x \in R \;\therefore\; D \ge 0 \Rightarrow (2 - y)^2 - 4 \times 1 \times 3 \ge 0$

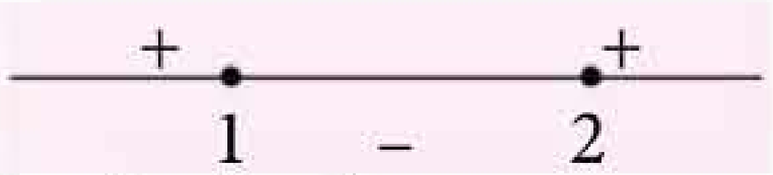
$\Rightarrow \; y \in \left(-\infty, 2 - 2\sqrt{3}\right] \cup \left[2 + 2\sqrt{3}, \infty\right)$

6. Let, $f(x) = y = \dfrac{x^2}{x^2+1} \Rightarrow x^2 = \dfrac{y}{1-y}$

$\Rightarrow \dfrac{y}{1-y} \ge 0 \qquad [\because x^2 \ge 0]$

$\therefore$ Range is $[0, 1)$.

7. To find the range of a polynomial function generally we use the concept of calculus.



$f'(x) = 6(x - 1)(x - 2)$

$\Rightarrow f(x)$ is increasing in $(-\infty, 1] \cup [2, \infty)$ and decreasing in $(1, 2)$.

$\therefore \; f(1) = 2 - 9 + 12 + 1 = 6$

$f(2) = 2(8) - 9(4) + 12(2) + 1 = 5$

$f(-1) = -2 - 9 - 12 + 1 = -22$

$f(3) = 2(27) - 9(9) + 12(3) + 1 = 10$

Hence, the range is $[-22, 10]$.

8. Given, $\log_{1/4} x \ge 1 \Rightarrow \log_{1/4} x \ge \log_{1/4} \dfrac{1}{4}$

We know, $\log_b a_1 \ge \log_b a_2$

$$\Rightarrow \begin{cases} a_1 \ge a_2, & \text{if } b > 1 \\ a_1 \le a_2, & \text{if } 0 < b < 1 \end{cases}$$

The base of log lying between 0 and 1 hence inequality sign will change, hence $x \le 1/4$. Also log$x$ is defined only when $x > 0$.

Hence, $x \in \left(0, \dfrac{1}{4}\right]$.

9. If $f(x)$ is a one-one function then $f(x_1) = f(x_2)$ will always be satisfied when $x_1 = x_2$ in the domain.

Hence we can write $f(x_1) - f(x_2) = (x_1 - x_2)\, g(x_1, x_2) = 0$ where $g(x_1, x_2)$ is some function in x_1 and x_2. Now if $g(x_1, x_2) = 0$ gives some solution which is different from $x_1 = x_2$ and which lies in the domain, then f is many-one else one-one.

Consider, $f(x_1) = f(x_2) \Rightarrow \dfrac{x_1+1}{x_1^2+x_1+1} = \dfrac{x_2+1}{x_2^2+x_2+1}$

One solution of this is obviously $x_1 = x_2$. Also we got a relation in x_1 and x_2 and for each value of x_1 in the domain we get a corresponding value of x_2 which may or may not be same as x_1.

If $x_1 = 1$ we get $x_2 = \dfrac{-1}{2} \ne x_1$, and both lie in the domain of f.

Hence, we have two different values x_1 and x_2 for which $f(x)$ has the same value. Hence f is many one.

10. Given function is an onto function. Hence, range should be equal to co-domain.

Given, $f(x) = x^2 + 2x + b = (x + 1)^2 + b - 1$. Hence range of $f(x)$ is $[b - 1, \infty)$.

Since, $f(x)$ is an onto function.

$\therefore \; b - 1 = 2 \Rightarrow b = 3$

11. First check the function for one-one and onto. If function is one-one and onto then find inverse using the identity $f(f^{-1}(x)) = x$.

Given, $f(x) = 2^{x(x-1)} \Rightarrow \log f(x) = x(x - 1)\log_e 2$

$\Rightarrow \dfrac{1}{f(x)} f'(x) = (2x - 1)\log_e 2$

$\Rightarrow f'(x) = 2^{x(x-1)} \log_e 2(2x - 1)$

Thus, $f(x)$ is an increasing function in $[1, \infty)$, therefore, $f(x)$ is a one-one function.

Also range of $f(x)$ is $[1, \infty)$ which is equal to co-domain. Hence, the function is also onto.

To find $f^{-1}(x)$:

Let f^{-1} be the inverse function of f, then by rule of identity $f(f^{-1}(x)) = x$, we have $2^{f^{-1}(x)(f^{-1}(x)-1)} = x$

$\Rightarrow \log_2 x = (f^{-1}(x))^2 - f^{-1}(x)$

$\Rightarrow f^{-1}(x) = \dfrac{1 \pm \sqrt{1 + 4\log_2 x}}{2}$

$\Rightarrow f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4\log_2 x}}{2}$

($\because 1 \le f^{-1}(x) < \infty \;\therefore\; 1 + 4\log_2 x \ge 1$ and $x \ge 1$ therefore only positive sign is acceptable)

12. **(i)** $f(x) = \dfrac{|x^2-1|(\cos x)}{1 + \log(e^x + e^{-x})}$

$f(-x) = \dfrac{|(-x)^2-1|(\cos(-x))}{1+\log(e^{-x}+e^{-(-x)})} = \dfrac{|x^2-1|(\cos x)}{1+\log(e^x+e^{-x})} = f(x)$

$\Rightarrow f(x)$ is even.

(ii) $f(-x)=\log\left(\frac{1-(-x)}{1+(-x)}\right)=\log\left(\frac{1+x}{1-x}\right)=-f(x)$

$\Rightarrow$ $f(x)$ is odd.

(iii) $f(-x)=\log(-x+\sqrt{1+(-x)^2})$

$=\log\left(\frac{(-x+\sqrt{1+x^2})(x+\sqrt{1+x^2})}{(x+\sqrt{1+x^2})}\right)$

$=\log\left(\frac{1}{x+\sqrt{1+x^2}}\right)=-f(x)$

$\Rightarrow$ $f(x)$ is odd.

(iv) $f(-x) = \sin(-x) - \cos(-x) = -\sin x - \cos x$

$\Rightarrow$ $f(x)$ is neither even nor odd.

13. Period of $\sin 4x$ is $\pi/2$ and period of $\tan 2x$ is $\pi/2$. Hence, period of $f(x)$ is $\pi/2$.

14. (b) : Given, $f(x + 1) - f(x) = 8x + 3$

$\Rightarrow$ $[b(x+1)^2+c(x+1)+d]-(bx^2+cx+d)=8x+3$

$\Rightarrow$ $(2b)x+(b+c)=8x+3$

$\Rightarrow$ $2b=8,\ b+c=3\ \Rightarrow b=4,\ c=-1$

15. (a) : We have, $\phi(x) = a^x\ \Rightarrow\ \phi(p) = a^p$

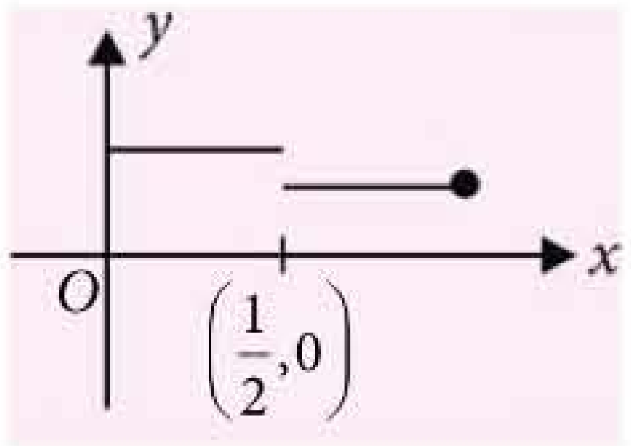
$\therefore$ $[\phi(p)]^3=[a^p]^3=a^{3p}=\phi(3p)$

16. (c) : $\frac{f(a)}{f(a+1)}=\frac{a/(a-1)}{(a+1)/a}=\frac{a^2}{a^2-1}=f(a^2)$

17. (d) : Here, $x^3 f\left(\frac{1}{x}\right)=x^3\left[\frac{4}{x^3}+\frac{3}{x^2}+\frac{3}{x}+4\right]$

$= 4 + 3x + 3x^2 + 4x^3 = f(x)$

18. (c) : The graph of the given function is



Thus, $f(x)$ is a step function.

19. (d) : Let $f(x)=\frac{x^2-1}{x^2+1}=\frac{x^2+1-2}{x^2+1}=1-\frac{2}{x^2+1}$

$\because$ $x^2+1\ge 1\Rightarrow \frac{2}{x^2+1}\le 2\ \Rightarrow 1-\frac{2}{x^2+1}\ge 1-2$

$\therefore$ $-1 \le f(x)$

Thus, $f(x)$ has the minimum value equal to -1.

20. (d) : $\because$ $y=\frac{ax+b}{cx-a}\ \Rightarrow\ x(cy-a)=b+ay$

$\Rightarrow$ $x=\frac{ay+b}{cy-a}=f(y)$

21. (d) : $\because$ $f(x) = |x|\ \Rightarrow\ f(-1) = f(1) = 1$

$\therefore$ $f(x)$ is many-one function. Obviously, f is not onto so f is neither one-one nor onto.

22. (c) : $f(x_1)=f(x_2)\Rightarrow x_1^2=x_2^2\Rightarrow x_1=x_2$, if $x\in R^+$

$\Rightarrow$ f is one-one. Since Range of $(f) = R^+ \subseteq R = Y$;

$\therefore$ f is not onto.

23. (b) : We have, $f(x) = (x - 1)(x - 2)(x - 3)$

$f(x)$ is not one-one as $f(1) = f(2) = f(3) = 0$.

For each $y \in R$, there exists $x \in R$ such that $f(x) = y$. Therefore f is onto. Hence $f : R \to R$ is onto but not one-one.

24. (c) : Function $f : R \to R$ is defined by $f(x) = e^x$. Let $x_1, x_2 \in R$ and $f(x_1) = f(x_2)$ or $e^{x_1} = e^{x_2}$ or $x_1 = x_2$. Therefore f is one-one.

Let $f(x) = e^x = y$.

Taking log on both sides, we get $x = \log y$.

In this function, negative real numbers have no pre-image so the function is not onto and zero is not the image of any real number. Therefore function f is into.

25. (b) : Domain of $f(x) = R - \{3\}$ and range $= \{1, -1\}$.

26. (c) : $f : N \to I$

$f(1) = 0, f(2) = -1, f(3) = 1, f(4) = -2, f(5) = 2,$
$f(6) = -3$ and so on.

In this type of function every element of set A has unique image in set B and there is no element left in set B. Hence, f is one-one and onto function.

27. (a) : Let $y=\sin^{-1}\left[\log_3\left(\frac{x}{3}\right)\right]\ \Rightarrow\ -1\le\log_3\left(\frac{x}{3}\right)\le 1$

$\Rightarrow$ $\frac{1}{3}\le\frac{x}{3}\le 3\ \Rightarrow 1\le x\le 9\ \Rightarrow\ x\in[1,9]$

28. (c) : For $x = -3$ and 3, $|x^2 - 9| = 0$

Therefore, $\log|x^2 - 9|$ does not exist at $x = -3, 3$

Hence, domain of function is $R - \{-3, 3\}$.

29. (b) : $x^2 - 6x + 7 = (x - 3)^2 - 2$

Obviously, minimum value is -2 and maximum is ∞. Hence, range of function is $[-2, \infty)$.

30. (b) : Obviously, here $|x| > 2$ and $x \neq 1$

i.e., $x \in (-\infty, -2) \cup (2, \infty)$

31. (d) : $1 + x \ge 0\ \Rightarrow\ x \ge -1$;

$1 - x \ge 0\ \Rightarrow\ x \le 1$; $x \neq 0$

Hence, domain is $[-1, 1] - \{0\}$.

32. (d) : $1-\frac{1}{x}>0\Rightarrow x>1$ and $x<0$. Also, $x \neq 0$

$\therefore$ Required interval $= (-\infty, 0) \cup (1, \infty)$

33. (d) : We have, $f(x)=\frac{(x-2)(x-1)}{(x-2)(x+3)}$

Hence, domain is $\{x : x \in R, x \neq 2, x \neq -3\}$

34. (a) : Here, $|x| > 1$, therefore $x \in (-\infty, -1) \cup (1, \infty)$.

35. (d) : Since maximum and minimum values of $\cos x - \sin x$ are $\sqrt{2}$ and $-\sqrt{2}$ respectively, therefore range of $f(x)$ is $[-\sqrt{2}, \sqrt{2}]$.

36. (b) : Given, $f(x) = \sin^2(x^4) + \cos^2(x^4) = 1$.
Hence range = $\{1\}$.

37. (c) : $f(x) = 1 + \dfrac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}$ $\Rightarrow$ Range = $(1, 7/3]$

38. (a) : Let $y = f(x) \Rightarrow y = \dfrac{2x+1}{1-3x} \Rightarrow y - 3xy = 2x + 1$

$$\Rightarrow x = \frac{y-1}{3y+2} \Rightarrow f^{-1}(y) = \frac{y-1}{3y+2} \Rightarrow f^{-1}(x) = \frac{x-1}{3x+2}$$

39. (b) : $(fog)(1) = f\{g(1)\} = f(\log 1) = e^0 = 1 = \log_e e$.

40. (a) : Here, $(gof)\left(\dfrac{-5}{3}\right) - (fog)\left(\dfrac{-5}{3}\right)$

$$= g\left\{f\left(\frac{-5}{3}\right)\right\} - f\left\{g\left(\frac{-5}{3}\right)\right\} = g(-2) - f\left(\frac{5}{3}\right) = 2 - 1 = 1$$

41. (d) : $(fog)(x) = f(g(x)) = f\left(\dfrac{x}{1-x}\right)$

$$= \frac{\frac{x}{1-x}}{\frac{x}{1-x} + 1} = \frac{x}{x + 1 - x} = x$$

42. (b) : $g(x) = 1 + \sqrt{x}$ and $f(g(x)) = 3 + 2\sqrt{x} + x$

$\Rightarrow \quad f(1 + \sqrt{x}) = 3 + 2\sqrt{x} + x$

Put $1 + \sqrt{x} = y \quad \Rightarrow x = (y-1)^2$

Then, $f(y) = 3 + 2(y-1) + (y-1)^2 = 2 + y^2$

Therefore $f(x) = 2 + x^2$.

43. (c) : $f : R \to R$, $f(x) = \sin x$ and $g : R \to R$, $g(x) = x^2$

$\Rightarrow fog(x) = f(g(x)) = f(x^2) = \sin x^2$

44. (b) : $(gof)(x) = |\sin x|$ and $f(x) = \sin^2 x$

$\Rightarrow \quad g(\sin^2 x) = |\sin x|; \therefore g(x) = \sqrt{x}$

45. (d)

46. (d) : $f(x) = \dfrac{\alpha x}{x+1}$; $f(f(x)) = f\left(\dfrac{\alpha x}{x+1}\right) = \dfrac{\alpha\left(\frac{\alpha x}{x+1}\right)}{\frac{\alpha x}{x+1} + 1}$;

But $f(f(x)) = x$, $\therefore \quad \dfrac{\alpha^2 x}{\alpha x + x + 1} = x$

Put $\alpha = -1$, $\dfrac{(-1)^2 x}{(-1)x + x + 1} = \dfrac{x}{-x + x + 1} = x \quad \therefore \alpha = -1$

47. (c) : It is well known fact that fractional part function always a periodic function whose period is 1.

48. (b) : The domain of function $\log_e(x - [x])$ is $R - Z$, because $x - [x] = \{x\}$ is a fractional part function defined over R and equals zero when $x \in Z$.

49. (d) : Here, $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
Two different numbers for numerator and denominator from these can be obtained in ${}^{10}P_2 = 10 \cdot 9 = 90$ ways

and if $\dfrac{p}{q} = \dfrac{q}{q} = 1$ (if numerator and denominator are same)

So, number of ways = 90 + 1 = 91.

50. (a) : Since the point $(-\beta, -\alpha)$ lies on $y = \log_e x$

$\Rightarrow \quad -\alpha = \log_e(-\beta) \quad \Rightarrow \quad -\beta = e^{-\alpha}$

$\Rightarrow \quad \beta = -e^{-\alpha} \quad \Rightarrow \quad y = -e^{-x}$

51. (b) : Statement-2 is true.

Now, $f(x) = 2 + \cos \Rightarrow f'(x) = -\sin x$

$\therefore$ Every interval of length π *i.e.*, $[t, t + \pi]$ contains at least one point, $0, \pm\pi, \pm 2\pi$.

$\therefore \quad f'(x) = 0$ in $[t, t + \pi]$; $t \in R$

$\therefore$ Statement-1 is true.

52. (c) : For $x \geq 0$, $f(x) = 0$

For $x < 0$, $f(x) = \sqrt{-2\sin^{-1} x}$

$\Rightarrow$ Range of $f(x)$ is $[0, \sqrt{\pi}]$.

and $f(x)$ is not an increasing function in its domain.

53. (d) : $f(-x) = g(-x) + g(x)$, $x > 0$

$\neq -g(x) - g(-x)$

54. (b) : Statement-2 is true.

Now $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{h \to 0} = \dfrac{\sin[0-h]}{[0-h]} = \lim\limits_{h \to 0} \dfrac{\sin(-1)}{(-1)} = \sin 1$

Also, $\lim\limits_{x \to 0^+} f(x) = 0$ because $[0 + h] = 0$.

Therefore $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$

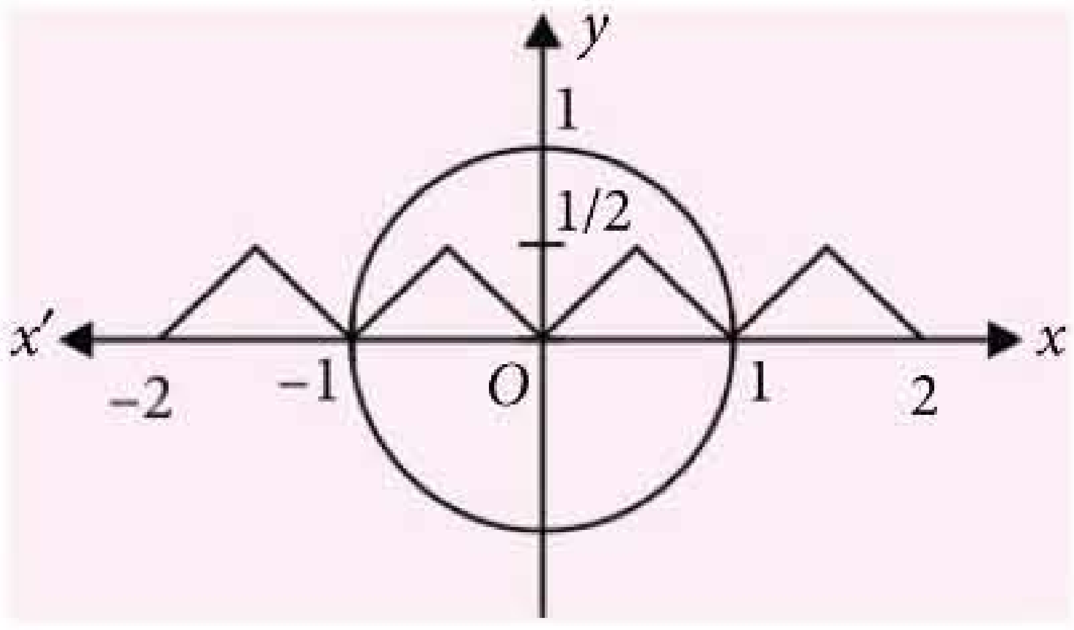
$\Rightarrow \quad \lim\limits_{x \to 0} f(x)$ does not exist.

55. (a) : We have, $\lim\limits_{x \to 0^+} \dfrac{x}{a}\left[\dfrac{b}{x}\right] = \lim\limits_{x \to 0^+} \dfrac{x}{a}\left(\dfrac{b}{x} - \left\{\dfrac{b}{x}\right\}\right)$

$$= \lim_{x \to 0^+}\left(\frac{b}{a} - \left(\frac{b}{a}\right)\frac{\{b/x\}}{b/x}\right) = \frac{b}{a} - 0 = \frac{b}{a}$$

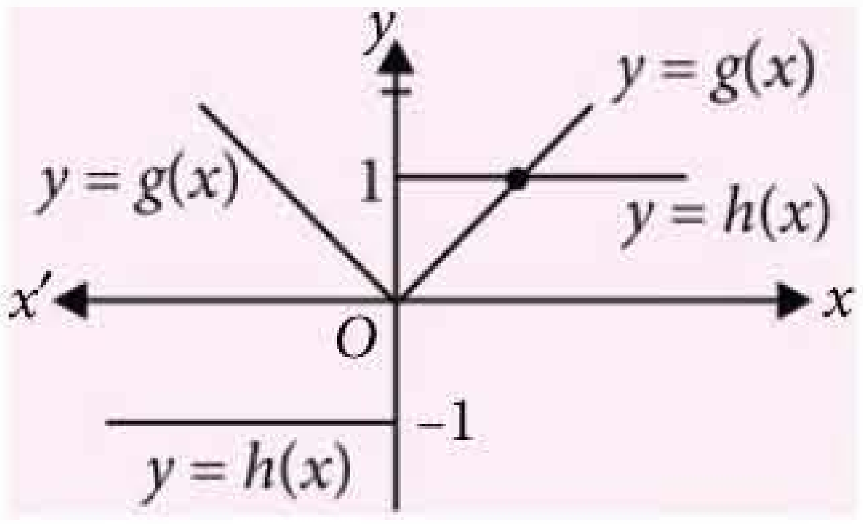
Thus, statement-1 is true and follows from statement-2.

56. (b) : Let $y = f(x) \Rightarrow x^2 + y^2 = 1$



∴ 2 solutions exists.

57. (a) : Let $y = g(x) = h(x)$



∴ Root of $g(x) - h(x)$ is 1.

58. (c) : $\int\limits_{-2}^{2} f(x)dx = 4\cdot\frac{1}{2}\times1\times\frac{1}{2}=1$

(59 - 61) : 59. (b) 60. (a) 61. (d)

Given $f(x+y) = \dfrac{f(x)+f(y)}{1-f(x)f(y)}$ ...(i)

Putting $x = y = 0$ in (i), we get

$$f(0)=\frac{f(0)+f(0)}{1-[f(0)]^2}$$

$\Rightarrow f(0)\,[f^2(0)+1] = 0 \Rightarrow f(0) = 0 \;\; (\because f^2(0) \neq -1)$

Now putting $y = -x$, we get

$$f(0)=\frac{f(x)+f(-x)}{1-f(x)\cdot f(-x)} \Rightarrow f(x)+f(-x)=0$$

$\Rightarrow f(-x) = -f(x) \Rightarrow f(x)$ is an odd function.

Now $f(y-x) = \dfrac{f(y)+f(-x)}{1-f(-x)\cdot f(y)}$

$$\Rightarrow \frac{f(y-x)}{y-x}=\frac{f(y)-f(x)}{1+f(x)\cdot f(y)}\cdot\frac{1}{(y-x)}$$

Taking limit as $y \to x$, we get

$$\lim_{y\to x}\frac{f(y-x)-f(0)}{y-x}=\lim_{y\to x}\frac{f(y)-f(x)}{y-x}\cdot\lim_{y\to x}\frac{1}{1+f(x)\cdot f(y)}$$

$$\Rightarrow f'(0)=f'(x)\cdot\frac{1}{1+f^2(x)} \Rightarrow f'(x)=f'(0)\,[1+f^2(x)]$$

$$\Rightarrow \frac{f'(x)}{f'(0)}=1+f^2(x)$$

62. A → p; B → p; C → q; D → s

$$f_a(x)=\frac{a^x}{a^x+\sqrt{a}} \Rightarrow f_a\left(\frac{1}{2}\right)=\frac{\sqrt{a}}{\sqrt{a}+\sqrt{a}}=\frac{1}{2}$$

$$\Rightarrow f_a(1-x)=\frac{a^{1-x}}{a^{1-x}+\sqrt{a}}=\frac{a/a^x}{(a/a^x)+\sqrt{a}}=\frac{\sqrt{a}}{a^x+\sqrt{a}}$$

$$\text{Now } f_a(x)+f_a(1-x)=\frac{a^x+\sqrt{a}}{a^x+\sqrt{a}}=1$$

(A) $$\sum_{k=1}^{1997} f_9\left(\frac{k}{1998}\right)=f_9\left(\frac{1}{1998}\right)+f_9\left(\frac{2}{1998}\right) +\ldots\ldots+f_9\left(\frac{1996}{1998}\right)+f_9\left(\frac{1997}{1998}\right)$$

$= t_1 + t_2 + t_3 + \ldots + t_{1997}$ ...(i)

Here $t_1 + t_{1997} = 1$, $t_2 + t_{1996} = 1$and so on.

Also, $t_{999}=f_9\left(\dfrac{999}{1998}\right)=f_9\left(\dfrac{1}{2}\right)=\dfrac{1}{2}$

∴ From (i), $\displaystyle\sum_{k=1}^{1997} f_9\left(\frac{k}{1998}\right)=1\times998+\frac{1}{2}=998.5$

(B) Similar to (A)

(C) Proceeding as in (A), we get

$$\sum_{k=1}^{2009} f_{16}\left(\frac{k}{2010}\right)=1\times1004+\frac{1}{2}=1004.5$$

(D) Proceeding as in (A), we get

$$\sum_{k=1}^{2008} f_{25}\left(\frac{k}{2009}\right)= 1\times1004=1004$$

63. A → s; B → r; C → p; D → q

(A) $L=\displaystyle\lim_{x\to-1}\frac{\sqrt[3]{7-x}-2}{x+1}=\lim_{h\to0}\frac{(8-h)^{1/3}-2}{h}$

(put $x + 1 = h$, as $x \to -1 \Rightarrow h \to 0$)

$$=\lim_{h\to0}\frac{2\left[1-\frac{h}{8}\right]^{\frac{1}{3}}-2}{h}=2\left[\lim_{h\to0}\frac{\frac{1}{3}\left(1-\frac{h}{8}\right)^{-2/3}\left(-\frac{1}{8}\right)}{1}\right]$$

$$=2\left(-\frac{1}{24}\right)=\frac{-1}{12} \Rightarrow 12L=-1$$

(B) $L=\displaystyle\lim_{x\to\frac{\pi}{4}}\left[\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}\right]$

$$=\lim_{x\to\frac{\pi}{4}}\left[\tan x\frac{(\tan x-1)(\tan x+1)}{\cos\left(x+\frac{\pi}{4}\right)}\right]$$

$$= \lim_{x\to\frac{\pi}{4}} \left[\frac{\tan x(\sin x - \cos x)(\tan x + 1)}{\cos x \cos\left(x + \frac{\pi}{4}\right)}\right]$$

$$= -\sqrt{2} \lim_{x\to\frac{\pi}{4}} \frac{\tan x\left[\cos x\frac{1}{\sqrt{2}} - \sin x\frac{1}{\sqrt{2}}\right](\tan x + 1)}{\cos x \cos\left(x + \frac{\pi}{4}\right)}$$

$$= -\sqrt{2} \lim_{x\to\frac{\pi}{4}} \frac{\tan x \cos\left(x + \frac{\pi}{4}\right)(\tan x + 1)}{\cos x \cos\left(x + \frac{\pi}{4}\right)}$$

$$= -\sqrt{2}\times 2\times\sqrt{2} = -4 \Rightarrow -(L/4) = 1$$

(C) $L = \lim\limits_{x\to 1} \dfrac{(2x-3)(\sqrt{x}-1)}{(2x+3)(\sqrt{x}+1)(\sqrt{x}-1)} = \dfrac{-1}{10}$

$\therefore \quad 20L = -2$

(D) $L = \lim\limits_{x\to\infty} \dfrac{\log x^n - [x]}{[x]} = \lim\limits_{x\to\infty} \dfrac{\log x^n}{[x]} - \lim\limits_{x\to\infty} \dfrac{[x]}{[x]}$

$= 0 - 1 = -1 \quad \therefore \quad -2L = 2$

64. A → p; B → r; C → s; D → q

(A) $f(x) = \begin{cases} x, & x > 1 \\ 0, & x = 1 \\ -x, & x < 1 \end{cases}$

So, $\lim\limits_{x\to 1} f(x)$ doesn't exist.

(B) $\lim\limits_{x\to 0} \dfrac{\sin(\sin(\tan(x^2/2)))}{\log\cos 3x}$

$$= \lim_{x\to 0} \frac{\sin(\sin(\tan(x^2/2)))}{\sin(\tan x^2/2)} \times \frac{\sin(\tan x^2/2)}{\tan x^2/2} \times \frac{\tan(x^2/2)}{x^2/2} \times \frac{x^2/2}{\log\cos 3x}$$

$$= \frac{1}{2}\lim_{x\to 0}\left(\frac{2x}{-3\tan 3x}\right) = -\frac{1}{9}$$

(C) $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+\tan^{-1}3x} - \sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x} - \sqrt{1+\tan^{-1}2x}}$

$$= \lim_{x\to 0} \frac{\left(1 + \frac{1}{3}\tan^{-1}3x + ...\right) - \left(1 - \frac{1}{3}\sin^{-1}3x + ...\right)}{\left(1 - \frac{1}{2}\sin^{-1}2x + ...\right) - \left(1 + \frac{1}{2}\tan^{-1}2x + ...\right)}$$

$$= \lim_{x\to 0} \frac{\frac{1}{3}\cdot\frac{\tan^{-1}3x}{x} + \frac{1}{3}\cdot\frac{\sin^{-1}3x}{x} + \frac{1}{x}(\text{higher power of } \sin^{-1}3x \text{ and } \tan^{-1}3x)}{-\frac{1}{2}\cdot\frac{\sin^{-1}2x}{x} - \frac{1}{2}\cdot\frac{\tan^{-1}2x}{x} + \frac{1}{x}(\text{higher power of } \sin^{-1}2x \text{ and } \tan^{-1}2x)}$$

$$= \frac{1+1}{-1-1} = -1$$

(D) $\lim\limits_{x\to 0^-} \dfrac{e^{1/x}-1}{e^{1/x}+1} = -1$ and $\lim\limits_{x\to 0^+} \dfrac{e^{1/x}-1}{e^{1/x}+1} = 1$

Thus, $\lim\limits_{x\to 0} f(x)$ doesn't exist.

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MATH archives

10 BEST PROBLEMS
$(x+y)^2 = x^2$

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of JEE (Main & Advanced) Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for JEE (Main & Advanced). In every issue of MT, challenging problems are offered with detailed solution. The reader's comments and suggestions regarding the problems and solutions offered are always welcome.

1. Among the following statements the number of correct statement(s) is/are

(I) For any two sets A and B, $A \cup (B - A) = A \cup B$.

(II) There exists three sets A, B and C such that $A \cap B$, $B \cap C$ and $A \cap C$ are non empty and $A \cap B \cap C$ is empty.

(III) Sets of letters needed to spell "CATARACT" and the set of letters needed to spell "TRACT" are equal sets.

(IV) Set A and B be two sets. If $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X, then $A = B$.

(a) 1 (b) 2 (c) 3 (d) 4

2. Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 5 elements and $B_1, B_2, \ldots, B_n$, are n sets each with 3 elements, let $\bigcup\limits_{i=1}^{30} A_i = \bigcup\limits_{j=1}^{n} B_j = S$ and each element of S belongs to exactly 10 of the $A_i's$ and exactly 9 of the $B_j's$. Then n is equal to

(a) 15 (b) 3
(c) 45 (d) None of these

3. Let A be the set of all determinants of order 3 with entries 0 or 1 only, B is the subset of A consisting of all determinants with value 1 and C is the subset consisting of all determinants with value -1. Then, if $n(B)$ and $n(C)$ denote the number of elements in B and C respectively, we have

(a) $C = \phi$ (b) $n(B) = n(C)$
(c) $A = B \cup C$ (d) $n(B) = 2n(C)$

4. The minimum number of elements that must be added to the relation $R = \{(1, 2), (2, 3)\}$ on the set of natural numbers, so that it is equivalence is

(a) 4 (b) 7 (c) 6 (d) 5

5. Let R be a relation on the set N be defined by $\{(x, y) | x, y \in N, 2x + y = 41\}$. Then, R is

(a) Reflexive (b) Symmetric
(c) Transitive (d) None of these

6. Consider any set of x_{101} observations $x_1, x_2, x_3, \ldots, x_{101}$. It is given that $x_1 < x_2 < x_3 < \ldots < x_{100} < x_{101}$. Then, the mean deviation of this set of observations about a point x_k is minimum, when k is

(a) 51 (b) 50 (c) 101 (d) 49

7. In an experiment with 10 observations on x, $\Sigma x = 60$, $\Sigma x^2 = 1000$. One observation that was 20 was found to be wrong and was replaced by the correct value 30, then the corrected variance is

(a) 64 (b) 111 (c) 52 (d) 101

8. In a series of 100 observations, half of them is equal to p and remaining half equal to $-p$. If the standard deviation of these observations is 10, then $|p| =$

(a) 10 (b) $\frac{1}{10}$ (c) $\frac{1}{\sqrt{10}}$ (d) $\sqrt{10}$

9. The median and S.D. of a distribution are 25 and 5 respectively. If each item is increased by 5, then the new median and S.D. will be

(a) 20, 10 (b) 30, 5 (c) 30, 10 (d) 25, 10

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10. Let $a_1, a_2, \ldots, a_n$ be n observations such that $\sum_{i=1}^{n} a_i = 50, \sum_{i=1}^{n} a_i^2 = 500$, then the possible value of n among the following is

(a) 3 (b) 5 (c) 4 (d) 2

SOLUTIONS

1. (d)

2. (c) : Since, $S = \bigcup_{i=1}^{30} A_i$

Since, element in the union S belongs to exactly 10 of the A_i's. So, $n(S) = \frac{1}{10}(5 \times 30) = 15$ ...(i)

Again, $S = \bigcup_{j=1}^{n} B_j$. So, $n(S) = \frac{1}{9}(3 \times n) = \frac{n}{3}$ ...(ii)

From (i) and (ii), $n/3 = 15 \Rightarrow n = 45$

3. (b) : Since, $-1 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \in C$

$\therefore$ C cannot be the empty set.

We also have, $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2$. So, $A \neq B \cup C$.

In general $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}$$

With all a's being 0 or 1 , Δ equals to 1 only if $a_{11}a_{22}a_{33} = 1$ and the remaining terms are zero or if $a_{12}a_{23}a_{31} = 1$ and the remaining terms are zero or if $a_{13}a_{21}a_{32} = 1$ and the remaining terms are zero.

Also, there are three similar relations for determinants that equals to -1, so we must have $n(B) = n(C)$.

4. (b) : Given, $R = \{(1, 2), (2, 3)\}$

Now, if R_1 is reflexive, then

$R_1 = \{(1, 2), (2, 3), (1, 1), (2, 2), (3, 3)\}$ has 5 elements.

If R_2 is both reflexive and symmetric, then $R_2 = \{(1, 2), (2, 3), (1, 1), (2, 2), (2, 1), (3, 2), (3, 3)\}$ has 7 elements.

Again, if R_3 is reflexive, symmetric and transitive all together, then $R_3 = \{(1, 2), (2, 3), (1, 1), (2, 2), (2, 1), (3, 2), (3, 3), (1, 3), (3, 1)\}$ has 9 elements.

$\therefore$ The minimum number of elements to be added is 7.

5. (d) : We have $R = \{(1, 39), (2, 37), (3, 35), (4, 33), (5, 31), (6, 29), (7, 27), (8, 25), (9, 23), (10, 21), (11, 19), (12, 17), (13, 15), (14, 13), (15, 11), (16, 9), (17, 7), (18, 5), (19, 3), (20, 1)\}$.

Clearly, R is not reflexive.

Also, $(1, 39) \in R$ but $(39, 1) \notin R$.

$\therefore$ R is not symmetric.

Now, $(15,11) \in R$ and $(11, 19) \in R$ but $(15, 19) \notin R$.

So, R is not transitive.

6. (a) : Given, $x_1 < x_2 < \ldots < x_{101}$

Median $= \left(\frac{102}{2}\right)^{\text{th}}$ item $= 51^{\text{th}}$ item. $\therefore$ x_{51} deviations will be minimum.

$\therefore$ Mean deviation will be minimum if $x_k = x_{51}$

$\Rightarrow k = 51$.

7. (d) : Given $\Sigma x = 60, \Sigma x^2 = 1000$

Increase in $\Sigma x = 10 \Rightarrow \Sigma x' = 60 + 10 = 70$

Increase in $\Sigma x^2 = 900 - 400 = 500$

$\Rightarrow \Sigma x'^2 = 1000 + 500 = 1500$

$$\text{Correct variance} = \frac{1}{n}\sum x'^2 - \left(\frac{1}{n}\sum x'\right)^2$$

$$= \frac{1}{10} \times 1500 - \left(\frac{1}{10} \times 70\right)^2 = 150 - 49 = 101$$

8. (a) : Since, mean $(\bar{x}) = 0$

$$\therefore \quad \text{S.D.} = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{100}} \Rightarrow 10 = \sqrt{\frac{100p^2}{100}} \Rightarrow |p| = 10$$

9. (b)

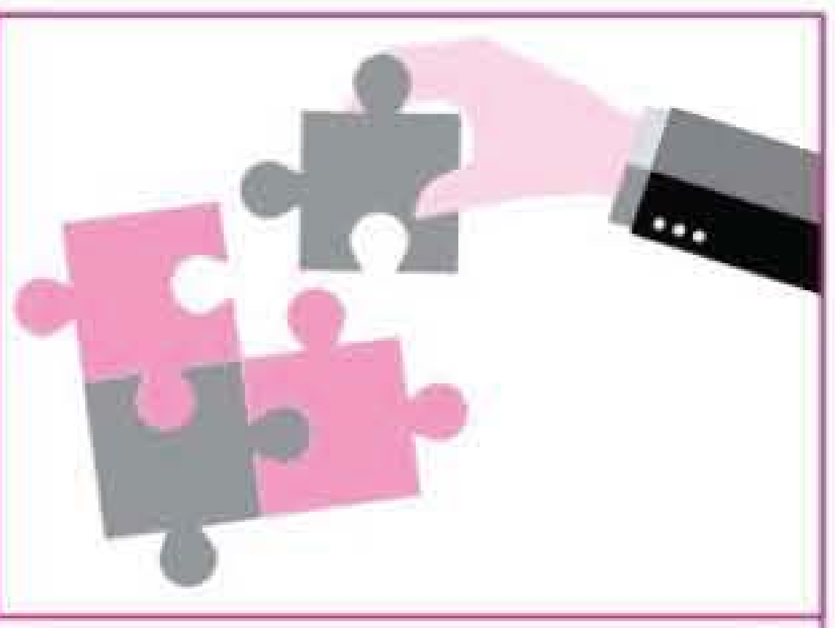
10. (b) : Since, $\frac{\sum a_i^2}{n} \geq \left(\frac{\sum a_i}{n}\right)^2 \Rightarrow \frac{500}{n} \geq \frac{2500}{n^2}$

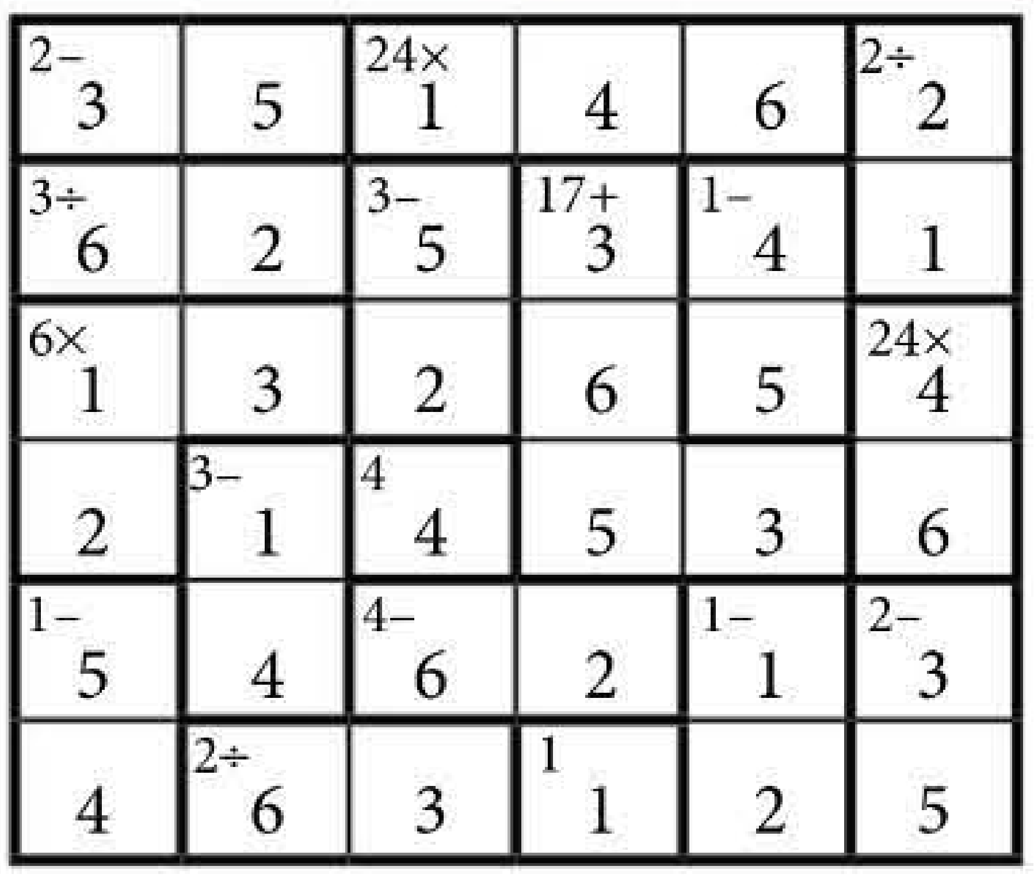
$\Rightarrow n \geq 5$

◆ ◆

PUZZLE CORNER

ANSWER - JULY 2019





2– 3	5	24× 1	4	6	2÷ 2
3÷ 6	2	3– 5	17+ 3	1– 4	1
6× 1	3	2	6	5	24× 4
2	3– 1	4 4	5	3	6
1– 5	4	4– 6	2	1– 1	2– 3
4	2÷ 6	3	1 1	2	5

CONCEPT MAP

COMPLEX NUMBERS

Class XI

Complex Number

A number of the form $z = a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$, is called a complex number. Here $\text{Re}(z) = a$ and $\text{Im}(z) = b$.

Algebra of Complex Numbers

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, then

- $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$
- $z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$
- $z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$
- $\dfrac{z_1}{z_2} = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{(a_2 b_1 - a_1 b_2)}{a_2^2 + b_2^2}$, where $z_2 \neq 0$
- $z_1 = z_2 \Rightarrow a_1 = a_2$ and $b_1 = b_2$

Geometry of Complex Numbers

If z is a variable point and z_1, z_2 are two fixed points in the argand plane, then

- $|z - z_1| = |z - z_2|$ represents perpendicular bisector of the line segment joining z_1 and z_2.
- $|z - z_1| + |z - z_2| = K$ (a fixed quantity > 0) ...(i)
  - If $K > |z_1 - z_2|$, then (i) represents an ellipse.
  - If $K = |z_1 - z_2|$, then (i) represents the line segment joining z_1 and z_2.
  - If $K < |z_1 - z_2|$, then (i) does not represent any curve in the argand plane.
  - If $K \neq |z_1 - z_2|$, then $|z - z_1| - |z - z_2| = K$ represent a hyperbola with foci at z_1 and z_2.
  - If $K = |z_1 - z_2|$, then $|z - z_1| - |z - z_2| = K$ represents a straight line joining z_1 and z_2 but excluding the line segment joining z_1 and z_2.
- Triangle ABC with vertices $A(z_1)$, $B(z_2)$ and $C(z_3)$ is equilateral if and only if $\begin{vmatrix} 1 & z_1 & z_2 \\ 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \end{vmatrix} = 0$.
- The equation of circle whose centre is at point having affix z_0 and radius R is $|z - z_0| = R$.
- The equation of circle whose centre is $-a$ and radius $R = \sqrt{|a|^2 - b}$ is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$.

Some Basic Terms

If $z = a + ib$, then

- Conjugate : $\bar{z} = a - ib$
- Modulus : $|z| = \sqrt{a^2 + b^2}$
- Argument : $\arg(z) = \tan^{-1}(b/a)$

Properties

- $\overline{(\bar{z})} = z$
- $z = \bar{z} \Leftrightarrow z$ is purely real
- $z + \bar{z} = 0 \Leftrightarrow z$ is purely imaginary
- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $\overline{(z_1 / z_2)} = \bar{z}_1 / \bar{z}_2; z_2 \neq 0$
- $\overline{(z^n)} = (\bar{z})^n$
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2 = |\bar{z}|^2$
- $|z^n| = |z|^n$, where $n \in Q$
- $|z_1 + z_2 + \ldots + z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2n\pi \ \forall \ n \in I$
- $\arg(z_1 / z_2) = \arg(z_1) - \arg(z_2) + 2n\pi \ \forall \ n \in I$
- $\arg(z^n) = n \arg(z) + 2n\pi \ \forall \ n \in I$

Different Forms of Complex Numbers

- **Polar Form :** $z = a + ib = r(\cos\theta + i\sin\theta) = r \text{ cis } \theta$, where $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}(b/a)$
- **Euler's Form :** $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$ where $-\pi < \theta < \pi$, θ is the principal argument.

Square Root of a Complex Number

Let $z = a + ib$ be a complex number So, square root of $z = a + ib$ is defined as,

$$\sqrt{a + ib} = \pm\left\{\sqrt{\frac{1}{2}\{\sqrt{a^2 + b^2} + a\}} + i\sqrt{\frac{1}{2}\{\sqrt{a^2 + b^2} - a\}}\right\}$$

To find the square root of $a - ib$, replace i by $-i$ in the above result.

Cube Roots of Unity

Let $x = (1)^{1/3} \Rightarrow x^3 - 1 = 0 \Rightarrow x = 1, \dfrac{-1 + i\sqrt{3}}{2}, \dfrac{-1 - i\sqrt{3}}{2}$

or $x = 1, \omega, \omega^2$

Note : (i) $\omega^3 = 1$ (ii) $1 + \omega + \omega^2 = 0$

(iii) $1 + \omega^n + \omega^{2n} = \begin{cases} 0, \text{ if } n \text{ is not a multiple of } 3 \\ 3, \text{ if } n \text{ is a multiple of } 3 \end{cases}$

DIFFERENTIABILITY

Class XII

CONCEPT MAP

Differentiability

A real valued function $f(x)$ is differentiable at $x = a$ if $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ exists finitely and it is denoted by $f'(a)$. In other words $f(x)$ is said to be differentiable at $x = a$ if its left hand derivative *i.e.*, $Lf'(a) = \lim\limits_{h \to 0} \dfrac{f(a - h) - f(a)}{-h}$ and right hand derivative *i.e.*, $Rf'(a) = \lim\limits_{h \to 0} \dfrac{f(a + h) - f(a)}{h}$ at $x = a$ are exist and equal.

Note : Every differentiable function is continuous but the converse is not necessarily true.

Derivative of Functions

- **Composite Function (Chain Rule) :** Let $y = f(t)$ and $t = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx}$.
- **Implicit Function :** Here we differentiate the function of type $f(x, y) = 0$.
- **Parametric Function :** If $x = f(t)$ and $y = g(t)$, then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{g'(t)}{f'(t)}$, $f'(t) \neq 0$
- **Logarithmic Differentiation :** If $y = u^v$, where u, v are functions of x, then $\log y = v \log u \Rightarrow \dfrac{d}{dx}(u^v) = u^v\left[\dfrac{v}{u}\dfrac{du}{dx} + \log u \dfrac{dv}{dx}\right]$

Second Order Derivative

Let $y = f(x)$, then $\dfrac{dy}{dx} = f'(x)$. If $f'(x)$ is differentiable, then $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(f'(x))$ or $\dfrac{d^2y}{dx^2} = f''(x)$ is the second derivative.

Leibnitz Formula

The n^{th} derivative of the product of two functions : If u, v be two functions possessing derivative of the n^{th} order, then

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots + {}^nC_r u_{n-r} v_r + \ldots + {}^nC_n u v_n$$

$$= \sum_{r=0}^{n} C(n, r) u_{n-r} v_r, \text{ where } {}^nC_r = \frac{n!}{(n-r)!r!}$$ and $C_0, C_1, \ldots\ldots, C_n$ are binomial coefficients.

Algebra of Derivatives

- $(u \pm v)' = u' \pm v'$
- $(uv)' = u'v + uv'$
- $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}, v \neq 0$
- $(\alpha u)' = \alpha u'$, where α is any scalar

Some Standard Derivatives

- $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$
- $\dfrac{d}{dx}(\cos^{-1} x) = \dfrac{-1}{\sqrt{1 - x^2}}$
- $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$
- $\dfrac{d}{dx}(\cot^{-1} x) = \dfrac{-1}{1 + x^2}$
- $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{|x|\sqrt{x^2 - 1}}$
- $\dfrac{d}{dx}(\text{cosec}^{-1} x) = \dfrac{-1}{|x|\sqrt{x^2 - 1}}$
- $\dfrac{d}{dx}(e^x) = e^x$
- $\dfrac{d}{dx}a^x = a^x \log a, a > 0$
- $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}, x > 0$
- $\dfrac{d}{dx}(\ln(x + \sqrt{x^2 + 1})) = \dfrac{1}{\sqrt{x^2 + 1}}$
- $\dfrac{d}{dx}(\ln(x + \sqrt{x^2 - 1})) = \dfrac{1}{\sqrt{x^2 - 1}}$
- $\dfrac{d^n}{dx^n}\cos(ax + b) = a^n \cos\left(ax + b + \dfrac{n\pi}{2}\right)$
- $\dfrac{d^n}{dx^n}\sin(ax + b) = a^n \sin\left(ax + b + \dfrac{n\pi}{2}\right)$
- $\dfrac{d^n}{dx^n}e^{ax}\cos(bx + c) = r^n e^{ax}\cos(bx + c + n\theta)$
- $\dfrac{d^n}{dx^n}e^{ax}\sin(bx + c) = r^n e^{ax}\sin(bx + c + n\theta)$, where $a = r\cos\theta, b = r\sin\theta, r = \sqrt{a^2 + b^2}, \theta = \tan^{-1}\dfrac{b}{a}$

Some Important Theorems

- **Rolle's Theorem :** If $f(x)$ is a real valued function defined on $[a, b]$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then $\exists$ atleast some $c \in (a, b)$ such that $f'(c) = 0$.
- **Lagrange's Mean Value Theorem :** If $f(x)$ is a real valued function defined on $[a, b]$ is continuous on $[a, b]$ and differentiable on (a, b), then $\exists$ atleast some $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

MOCK TEST PAPER 2020

JEE MAIN

Time : 1 hr 15 min.

Series - 3

The entire syllabus of Mathematics of JEE-MAIN is being divided in to eight units, on each unit there will be a Mock Test Paper (MTP) which will be published in the subsequent issues. The syllabus for module break-up is given bellow:

	Topic	Syllabus In Detail
UNIT NO. 3	Permutations & Combinations	Fundamental principle of counting, permutation as an arrangement and combination as selection, meaning of P (n, r) and C (n, r), simple applications.
	Trigonometry	General solution and Properties of Triangle.
	Co-ordinate geometry-2D	Circles : Standard form of equation of a circle, general form of the equation of a circle, its radius and centre, equations of a circle when the end points of a diameter are given, points of intersection of a line and a circle with the centre at the origin and condition for a line to be tangent to a circle, equation of the tangent.

1. If ${}^{n-1}C_r = (k^2 - 3)\,{}^{n}C_{r+1}$, then $k \in$

(a) $(-\infty, -2]$ (b) $[2, \infty)$

(c) $(-\sqrt{3}, \sqrt{3})$ (d) $(\sqrt{3}, 2]$

2. The number of zeros at the end of 70! is

(a) 16 (b) 5 (c) 7 (d) 70

3. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in English dictionary. The number of words that appear before the word COCHIN, is

(a) 360 (b) 192 (c) 96 (d) 48

4. The number of signals that can be generated by using 6 differently coloured flags, when any number of them may be hoisted at a time is

(a) 1956 (b) 1957 (c) 1958 (d) 1959

5. Seven different lecturers are to deliver lectures in seven periods of a class on a particular day. P, Q and R are three of the lecturers. The number of ways in which a routine for the day can be made such that P delivers his lectures before Q and Q before R, is

(a) 420 (b) 120

(c) 210 (d) None of these

6. Let A be the set of all positive prime integer less than 30. The number of different rational numbers whose numerator and denominator belong to A is

(a) 90 (b) 180

(c) 91 (d) None of these

7. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly 4 girls consecutively in the queue. Then, the value of $\frac{m}{n}$ is

(a) 4 (b) 5 (c) 6 (d) 7

8. The number of ordered triplets of positive integers which are solutions of the equation $x + y + z = 100$ is

(a) 6005 (b) 4851 (c) 5081 (d) 5151

9. The total number of ways in which 5 balls of different colours can be distributed among 3 persons, so that each person get at least one ball, is

(a) 75 (b) 150 (c) 210 (d) 243

10. In a certain test there are n questions. In this test 2^{n-k} students gave wrong answers to at least k questions, where $k = 1, 2, 3, \ldots.\ n$. If the total number of wrong answers given is 2047, then n is equal to

(a) 10 (b) 11 (c) 12 (d) 13

11. The general solution of the equation $\sin 2x + 2\sin x + 2\cos x + 1 = 0$ is

By : Sankar Ghosh, S.G.M.C, Mob. 09831244397.

(a) $3n\pi-\frac{\pi}{4}, n\in Z$ (b) $2n\pi+\frac{\pi}{4}, n\in Z$

(c) $2n\pi+(-1)^n\sin^{-1}\left(\frac{1}{\sqrt{3}}\right), n\in Z$

(d) $n\pi-\frac{\pi}{4}, n\in Z$

12. The number of solutions of the equation $2\sin^3 x+2\cos^3 x-3\sin 2x+2=0$ in $[0, 4\pi]$ is

(a) 2 (b) 3 (c) 4 (d) 5

13. If $\cos 2\theta$, 1 and $\sin\theta$ are in G.P., then $\theta =$

(a) $n\pi+(-1)^n\frac{\pi}{2}, n\in Z$

(b) $n\pi+(-1)^{n-1}\frac{\pi}{2}, n\in Z$

(c) $2n\pi, n\in Z$ (d) None of these

14. The most general solutions of the equation $2^{\sin x}+2^{\cos x}=2^{1-\frac{1}{\sqrt{2}}}$ is given by

(a) $x=n\pi-\frac{\pi}{4}, n\in Z$ (b) $x=n\pi+\frac{\pi}{4}, n\in Z$

(c) $x=n\pi+(-1)^n\frac{\pi}{4}, n\in Z$

(d) $x=2n\pi\pm\frac{\pi}{4}, n\in Z$

15. The solution set of number of values of $\sin\left(x-\frac{\pi}{4}\right)-\cos\left(x+\frac{3\pi}{4}\right)=1$ and $\frac{2\cos 7x}{\cos 3+\sin 3}>2^{\cos 2x}$ is

(a) $k\pi+(-1)^k\frac{\pi}{4}, k\in Z$ (b) $(8k+3)\frac{\pi}{4}, k\in Z$

(c) $(8k+1)\frac{\pi}{4}, k\in Z$ (d) None of these

16. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2-c^2=y$, where c is the third side of the triangle, then the ratio of the inradius to the circumradius of the triangle is

(a) $\frac{3y}{2x(x+c)}$ (b) $\frac{3y}{2c(x+c)}$

(c) $\frac{3y}{4x(x+c)}$ (d) $\frac{3y}{4c(x+c)}$

17. In a ΔABC, if $2b=a+c$ and $A-C=90°$, then $\sin B=$

(a) $\frac{\sqrt{7}}{5}$ (b) $\frac{\sqrt{5}}{8}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{5}}{3}$

18. Let ABC be a triangle such that $\angle ACB=\frac{\pi}{6}$ and let a, b and c denote the triangle of the sides opposite to A, B and C respectively. The value of x for which $a=x^2+x+1$, $b=x^2-1$ and $c=2x+1$ is

(a) $-(2+\sqrt{3})$ (b) $(1+\sqrt{3})$

(c) $2+\sqrt{3}$ (d) $4\sqrt{3}$

19. If the angles A, B and C of a triangle are in arithmetic progression and if a, b and c denote the lengths of the sides opposite to angles A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A$ is

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\sqrt{3}$

20. If a chord AB of a circle subtends an angle $\theta\left(\neq\frac{\pi}{3}\right)$ at a point C on the circumference such that the triangle ABC has maximum area, then

(a) $A=\frac{\pi}{3}+\frac{\theta}{2}, B=\frac{2\pi}{3}-\frac{3\theta}{2}$

(b) $A=\frac{\pi}{4}+\frac{\theta}{2}, B=\frac{2\pi}{4}-\frac{3\theta}{2}$

(c) $A=\frac{\pi}{6}+\theta, B=\frac{5\pi}{6}+2\theta$

(d) None of these

21. The centre of the circle inscribed in the square formed by the lines $x^2-8x+12=0$ and $y^2-14y+45=0$ is

(a) (4, 7) (b) (7, 4) (c) (9, 4) (d) (4, 9)

22. A circle of radius 2 units touches the coordinate axes in the first quadrant. If the circles makes a complete rotation on the x-axis along the positive direction of the x-axis, then the equation of the circle in the new position is

(a) $x^2+y^2-4(x+y)-8\pi x+(4\pi+2)^2=0$

(b) $x^2+y^2-4(x+y)+(4\pi+2)^2=0$

(c) $x^2+y^2-8\pi x-4y+(4\pi+2)^2=0$

(d) None of these

23. The equation of the circle passing through (1, 0) and (0, 1) and having the smallest possible radius, is

(a) $x^2+y^2+x+y-2=0$

(b) $x^2+y^2=x+y$

(c) $x^2+y^2=1$ (d) None of these

24. The set of values of a for which the point $(2a, a+1)$ is an interior point of the larger segment of the circle $x^2+y^2-2x-2y-8=0$ made by the chord $x-y+1=0$, is

(a) $\left(-1,\frac{9}{5}\right)$ (b) $\left(0,\frac{9}{5}\right)$

(c) $(0, \infty)$ (d) None of these

25. Let L_1 be a straight line passing through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circles $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then which of the following equations can represent L_1?

(a) $x + y = 0, x - 7y = 0$ (b) $x - y = 0, x + 7y = 0$

(c) $7x + y = 0$ (d) $x - 7y = 0$

26. If the tangent from a point P to the circle $x^2 + y^2 = 1$ is perpendicular to the tangent from P to the circle $x^2 + y^2 = 3$, then the locus of P is

(a) a circle of radius 2 (b) a circle of radius 4

(c) a circle of radius 3 (d) None of these

27. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$ is

(a) 0 (b) 1 (c) 3 (d) 4

28. The locus of the centre of a circle which cuts orthogonally the circle $x^2 + y^2 - 20x + 4 = 0$ and which touches $x = 2$, is

(a) $y^2 = 16x + 4$ (b) $x^2 = 16y$

(c) $x^2 = 16y + 4$ (d) $y^2 = 16x$

29. There are two circles whose equations are $x^2 + y^2 = 9$ and $x^2 + y^2 - 8x - 6y + n^2 = 0, n \in Z$, having exactly two common tangents. The number of possible values of n is

(a) 2 (b) 8

(c) 9 (d) None of these

30. A triangle PQR is inscribed in the circle $x^2 + y^2 = 25$. If coordinates of Q and R are $(3, 4)$ and $(-4, 3)$ respectively, then $\angle QPR =$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

SOLUTIONS

1. (d) : We have, ${}^{n-1}C_r = (k^2 - 3)\,{}^nC_{r+1}$

$$\Rightarrow \frac{(n-1)!}{r!(n-1-r)!} = (k^2 - 3)\frac{n!}{(r+1)!(n-r-1)!}$$

$$\Rightarrow \frac{(n-1)!}{r!(n-1-r)!} = (k^2 - 3)\frac{n(n-1)!}{(r+1)r!(n-r-1)!}$$

$$\Rightarrow 1 = \frac{(k^2-3)n}{(r+1)}$$

$$\Rightarrow \frac{(r+1)}{n} = k^2 - 3 \quad [\because\ 0 \le r \le n-1 \Rightarrow r+1 \le n]$$

$\Rightarrow k^2 - 3 \le 1 \Rightarrow k^2 - 4 \le 0 \Rightarrow -2 \le k \le 2$ ...(i)

Again, ${}^{n-1}C_r = (k^2 - 3)\,{}^nC_{r+1}$

$$\Rightarrow k^2 - 3 = \frac{{}^{n-1}C_r}{{}^nC_{r+1}} > 0 \quad \Rightarrow \quad k^2 - 3 > 0$$

$\Rightarrow k < -\sqrt{3}$ or $k > \sqrt{3}$ ...(ii)

From (i) and (ii), we have $k \in (\sqrt{3}, 2]$

2. (a) : We have, $70! = 2^a \times 3^b \times 5^c \times 7^d \times ...$

$$\text{Now, } a = E_2(70!) = \left[\frac{70}{2}\right] + \left[\frac{70}{2^2}\right] + \left[\frac{70}{2^3}\right] + \left[\frac{70}{2^4}\right] + \left[\frac{70}{2^5}\right] + \left[\frac{70}{2^6}\right]$$

$\Rightarrow a = 35 + 17 + 8 + 4 + 2 + 1 = 67$

$$\text{and } c = E_5(70!) = \left[\frac{70}{5}\right] + \left[\frac{70}{5^2}\right] = 14 + 2 = 16$$

$\therefore$ $70! = (10)^{16} \times 2^{51} \times 3^b \times 7^d \times .....$

Thus, the number of zeros at the end of 70! is 16.

3. (c) : The number of words starting with CC = 4!

The number of words starting with CH = 4!

The number of words starting with CI = 4!

The number of words starting with CN = 4!

$\because$ COCHIN is the first word in the list of words beginning with CO.

$\therefore$ Number of words that appear before the word COCHIN = 96.

4. (a) : The signals can be made by using at a time one or more flags.

The total number of signals when r flags are used at a time from 6 flags is equal to the number of arrangements of 6, taking r at a time, *i.e.*, ${}^6P_r$, where $r = 1, 2, 3, 4, 5, 6$. Hence, the number of different signals that can be generated is $\sum_{r=1}^{6} {}^6P_r$

$$= {}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5 + {}^6P_6 = 1956.$$

5. (d) : Let us consider the arrangements of 7 things out of which three are identical and remaining are distinct. In each arrangement of these 7 things, let us mark the first identical item as P, second as Q and third as R. We find that in all the $\frac{7!}{3!}$ arrangements, P will occur prior to Q and Q will occur prior to R.

$\therefore$ The required number of ways $= \frac{7!}{3!} = 840$

6. (a) : We have, $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$

Now, the number of different rational numbers whose numerator and denominator belong to A, is the number of arrangements of 10 members of the set A, taking 2 at a time.

$\therefore$ The required number of rational numbers $= {}^{10}P_2 = 90$.

7. (b) : Let us consider the 5 girls as a single individual and this single individual with 5 boys can be arranged in 6! ways. Also these 5 girls can arranged among themselves in 5! ways.

$\therefore$ Total numbers of arrangements in which 5 girls are always stand consecutively in the queue is $n = 6! \times 5!$

Now, we choose 4 girls from 5, to find the number of ways in which exactly 4 girls stand consecutively *i.e.*, ${}^5C_4$ ways. These 4 girls can stand in a queue in 4! ways. Now, we consider these 4 girls as a single individual and mixing with the remaining girl and 5 boys , the 7 individuals can be arranged in 7! ways. Thus number of ways in which four girls consecutively in a queue $= {}^5C_4 \times 4! \times 7!$

These ways also include the ways in which all five girls stand consecutively in a queue the number of such ways is $2({}^5C_4 \times 4! \times 6!)$

$\therefore$ Number of ways in which exactly 4 girls stand in a queue is $m = {}^5C_4 \times 4! \times 7! - 2({}^5C_4 \times 4! \times 6!)$

$$= {}^5C_4 \times 4! \times 6! \times 5 = 5 \times 5! \times 6!$$

$$\therefore \frac{m}{n} = \frac{5 \times 5! \times 6!}{5! \times 6!} = 5.$$

8. (d) : The number of solutions of the given equation is same as the number of ways of distributing 100 items among 3 persons such that each person can receive any number of items.

Hence, total number of solutions $= {}^{100+3-1}C_{3-1}$

$$= {}^{102}C_2 = \frac{102 \times 101}{2} = 51 \times 101 = 5151$$

Thus, the number of ordered triplets = 5151

9. (b) : Required number of ways = (Total number of ways in which 5 balls of different colours can be distributed among 3 persons) – (Number of ways in which at least one person gets no ball)

$= 3^5 - ({}^3C_1 \times 2^5 - {}^3C_2 \times 1^5) = 243 - 96 + 3 = 150$

10. (b) : The number of students answering exactly $k(1 \le k \le n-1)$questions wrongly is $2^{n-k} - 2^{n-k-1}$. The number of students answering all questions wrongly is 2^0. Thus, the total number of wrong answers

$$= 1 \cdot (2^{n-1} - 2^{n-2}) + 2 \cdot (2^{n-2} - 2^{n-3}) + \ldots + (n-1)(2^1 - 2^0) + n \cdot 2^0$$

$$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2^0 = 2^n - 1$$

Since, $2^n - 1 = 2047$ (Given)

$\Rightarrow 2^n = 2048 = 2^{11} \Rightarrow n = 11.$

11. (d) : The given equation is

$\sin 2x + 2\sin x + 2\cos x + 1 = 0$

$\Rightarrow 1 + \sin 2x + 2\sin x + 2\cos x = 0$

$\Rightarrow (\sin x + \cos x)^2 + 2(\sin x + \cos x) = 0$

$\Rightarrow (\sin x + \cos x)(\sin x + \cos x + 2) = 0$

$\Rightarrow (\sin x + \cos x) = 0$

$$\begin{bmatrix} \because\ 2 - \sqrt{2} \le \sin x + \cos x + 2 \le 2 + \sqrt{2} \\ \therefore\ \sin x + \cos x + 2 \ne 0 \end{bmatrix}$$

$$\therefore \sin x = -\cos x \Rightarrow \tan x = -1 = \tan\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow x = n\pi - \frac{\pi}{4}, n \in Z$$

12. (c) : Given, $2\sin^3 x + 2\cos^3 x - 3\sin 2x + 2 = 0$

$\Rightarrow \sin^3 x + \cos^3 x - 3\sin x\cos x + 1 = 0$

$\Rightarrow \sin^3 x + \cos^3 x + 1 - 3\sin x\cos x = 0$

$\Rightarrow \sin x + \cos x + 1 = 0 \quad \Rightarrow \sin x + \cos x = -1$

$$\Rightarrow \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = -1$$

$$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} = \cos\left(\frac{3\pi}{4}\right)$$

$$\Rightarrow x - \frac{\pi}{4} = 2n\pi \pm \frac{3\pi}{4} \Rightarrow x = 2n\pi \pm \frac{3\pi}{4} + \frac{\pi}{4}, \ n \in Z$$

$$\Rightarrow x = 2n\pi + \pi, 2n\pi - \frac{\pi}{2}$$

$$\Rightarrow x = (2n+1)\pi, (4n-1)\frac{\pi}{2}$$

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When $n = 0$, then $x = \pi, -\dfrac{\pi}{2}$;

When $n=1$, then $x=3\pi, \dfrac{3\pi}{2}$; When $n=2$ then $x=5\pi, \dfrac{7\pi}{2}$

Hence, in $[0, 4\pi]$ the solutions are $\pi, 3\pi, \dfrac{3\pi}{2}, \dfrac{7\pi}{2}$.

13. (b) : Given that $\cos 2\theta$, 1 and $\sin\theta$ are in G.P.

$\therefore \quad 1 = \sin\theta \cos 2\theta$

$\Rightarrow \quad \sin\theta(1 - 2\sin^2\theta) = 1 \Rightarrow 2\sin^3\theta - \sin\theta + 1 = 0$

$\Rightarrow \quad (\sin\theta + 1)(2\sin^2\theta - 2\sin\theta + 1) = 0$

$\Rightarrow \quad \sin\theta + 1 = 0 \qquad [\because \; 2\sin^2\theta - 2\sin\theta + 1 \neq 0]$

$$\Rightarrow \sin\theta = -1 = \sin\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow \theta = n\pi + (-1)^n\left(-\frac{\pi}{2}\right) = n\pi + (-1)^{n-1}\left(\frac{\pi}{2}\right), n \in Z$$

14. (b) : We know that

$$\frac{2^{\sin x} + 2^{\cos x}}{2} \geq \sqrt{2^{\sin x + \cos x}} \;\; \forall\, x \in R \;\; [\because \text{A.M.} \geq \text{G.M.}]$$

$$\Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2^{1+\frac{1}{2}(\sin x + \cos x)} \quad \forall\, x \in R$$

$$\Rightarrow 2^{\sin x} + 2^{\cos x} \geq 2^{1-\frac{1}{\sqrt{2}}} \; [\because -\sqrt{2} \leq \sin x + \cos x \leq \sqrt{2}]$$

Thus, $2^{\sin x} + 2^{\cos x} = 2^{1-\frac{1}{\sqrt{2}}}$ holds only when, $2^{\sin x} = 2^{\cos x}$

$$\Rightarrow \sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}, n \in Z$$

15. (b) : We have, $\sin\left(x - \dfrac{\pi}{4}\right) - \cos\left(x + \dfrac{3\pi}{4}\right) = 1$

$$\Rightarrow \sin x = \frac{1}{\sqrt{2}} \Rightarrow x = 2n\pi + \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}, n \in Z$$

For these values of x, we have $\cos 2x = 0$.

Also, $\cos 3 + \sin 3 < 0$

$$\therefore \frac{2\cos 7x}{\cos 3 + \sin 3} > 2^{\cos 2x} \Rightarrow \frac{2\cos 7x}{\cos 3 + \sin 3} > 1 \Rightarrow \cos 7x < 0$$

Clearly, $x = 2n\pi + \dfrac{3\pi}{4}, n \in Z$ satisfy this equation.

16. (b) : Let the sides of the triangle are a, b and the third side be c. Therefore, we have

$a + b = x, ab = y$ and $x^2 - c^2 = y$

$\Rightarrow \quad (a + b)^2 - c^2 = ab \Rightarrow a^2 + b^2 - c^2 = -ab$

$$\Rightarrow \frac{a^2 + b^2 - c^2}{2ab} = -\frac{1}{2} \Rightarrow \cos C = -\frac{1}{2} = \cos\frac{2\pi}{3}$$

Let Δ be the area of the ΔABC.

$$\therefore \quad \Delta = \frac{1}{2}ab\sin C = \frac{1}{2}y \times \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{4}y$$

Let r and R be the respectively inradius and circumradius of ΔABC.

Then, $r = \dfrac{\Delta}{s}$ and $R = \dfrac{abc}{4\Delta}$

$$\therefore \quad \text{Required ratio} = \frac{r}{R} = \frac{4\Delta^2}{s(abc)} = \frac{4\left(\frac{\sqrt{3}}{4}y\right)^2}{\left(\frac{x+c}{2}\right)(yc)}$$

$$\Rightarrow \frac{r}{R} = \frac{3y}{2c(x+c)}$$

17. (c) : We are given that $2b = a + c$

$\Rightarrow \quad 4R\sin B = 2R\sin A + 2R\sin C$

$\Rightarrow \quad 2\sin B = \sin A + \sin C$

$$\Rightarrow 2\left(2\sin\frac{B}{2}\cos\frac{B}{2}\right) = 2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)$$

$$\Rightarrow 4\sin\frac{B}{2}\cos\frac{B}{2} = 2\cos\frac{B}{2}\cos\left(\frac{A-C}{2}\right)$$

$$\Rightarrow 2\sin\frac{B}{2} = \cos\left(\frac{A-C}{2}\right)$$

$$\Rightarrow 2\sin\frac{B}{2} = \cos 45^\circ \; \left[\because \; A - C = 90^\circ \text{ (Given)}\right]$$

$$\Rightarrow \sin\frac{B}{2} = \frac{1}{2\sqrt{2}} \Rightarrow \cos\frac{B}{2} = \frac{\sqrt{7}}{2\sqrt{2}}$$

Hence, $\sin B = 2\sin\dfrac{B}{2}\cos\dfrac{B}{2} = \dfrac{\sqrt{7}}{4}$

18. (b) : We know that, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$

$$\Rightarrow \cos\frac{\pi}{6} = \frac{(x^2 + x + 1)^2 + (x^2 - 1)^2 - (2x + 1)^2}{2(x^2 + x + 1)(x^2 - 1)}$$

$$\Rightarrow \sqrt{3} = \frac{2x^2 + 2x - 1}{x^2 + x + 1}$$

$$\Rightarrow (2 - \sqrt{3})x^2 + (2 - \sqrt{3})x - (\sqrt{3} + 1) = 0$$

$$\Rightarrow x = \frac{-(2 - \sqrt{3}) \pm \sqrt{(2 - \sqrt{3})^2 + 4(2 - \sqrt{3})(\sqrt{3} + 1)}}{2(2 - \sqrt{3})}$$

$$\Rightarrow x = -(2 + \sqrt{3}), (\sqrt{3} + 1) \Rightarrow x = (\sqrt{3} + 1)$$

19. (d) : Given that, A, B, C are in A.P.

$\therefore \quad 2B = A + C \Rightarrow 2B + B = A + C + B$

$\Rightarrow \quad 3B = 180^\circ \Rightarrow B = 60^\circ$

Now, $\dfrac{a}{c}\sin 2C + \dfrac{c}{a}\sin 2A$

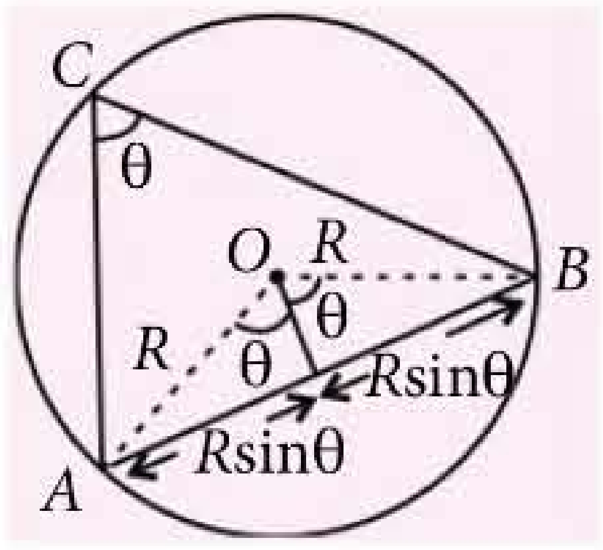
$$= \frac{2a}{c}\sin C\cos C + \frac{2c}{a}\sin A\cos A$$

$$= \frac{2(2R\sin A)\sin C\cos C}{2R\sin C} + \frac{2(2R\sin C)\sin A\cos A}{2R\sin A}$$

$= 2(\sin A\cos C + \cos A\sin C)$

$= 2\sin(A + C) = 2\sin(\pi - B) = 2\sin B = 2\sin 60° = \sqrt{3}$

20. (d) : We have,



Δ = Area of ΔABC

$$= \frac{1}{2}(AB \times AC \times \sin A)$$

$$\Rightarrow \Delta = \frac{1}{2}[2R\sin\theta \times 2R\sin B \times \sin A]$$

$[\because AB = 2R\sin\theta \text{ and } AC = 2R\sin B]$

$\Rightarrow \Delta = 2R^2 \sin A\sin B\sin\theta$

$\Rightarrow \Delta = R^2\sin\theta\,[\cos(A - B) - \cos(A + B)]$

$\Rightarrow \Delta = R^2\sin\theta\,[\cos(\pi - 2B - \theta) - \cos(\pi - \theta)]$

$\Rightarrow \Delta = R^2\sin\theta\,[\cos\theta - \cos(2B + \theta)]$

Clearly, Δ is maximum, when $\cos(2B + \theta) = -1$

$$\Rightarrow 2B + \theta = \pi \quad \Rightarrow B = \frac{\pi}{2} - \frac{\theta}{2}$$

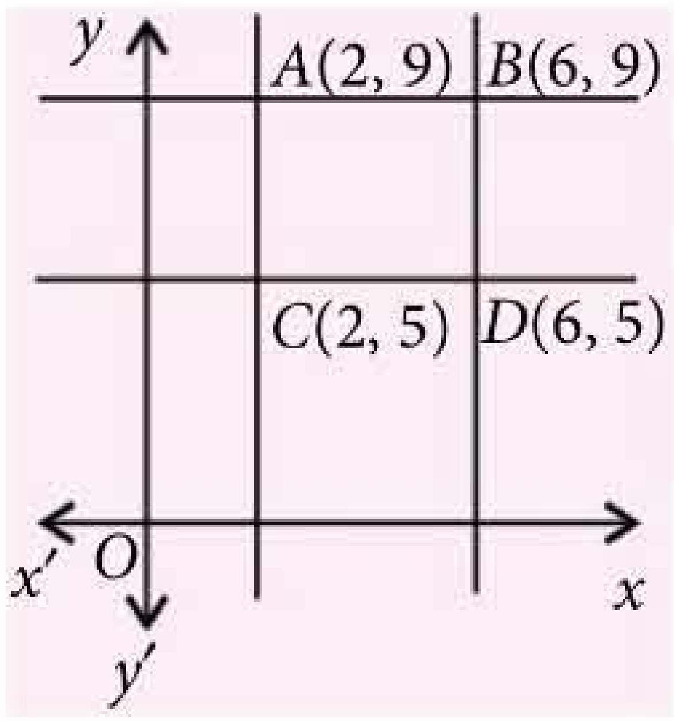
$$\therefore \quad A + B + \theta = \pi \Rightarrow A = \frac{\pi}{2} - \frac{\theta}{2}$$

Hence, area is maximum, when $A = B = \dfrac{\pi}{2} - \dfrac{\theta}{2}$

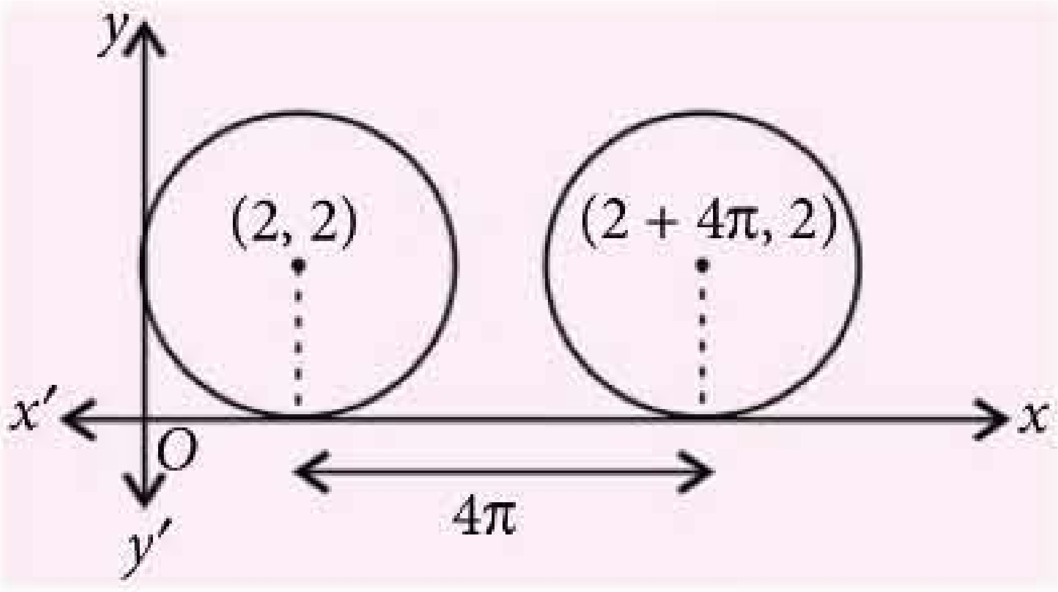
21. (a) : The equations of the sides of the square $ABCD$ are $x = 2$, $x = 6$, $y = 5$ and $y = 9$. The coordinates of the end points of a diameter are (2, 5) and (6, 9).

Therefore, the centre of the square is

$$\left(\frac{2+6}{2}, \frac{5+9}{2}\right) = (4, 7).$$



22. (a) : The centre and radius of the circle before rotation are (2, 2) and 2 respectively. Now, after the complete rotation of the circle along the positive direction of x-axis, the centre of the circle becomes $(2 + 4\pi, 2)$ and the radius remains same.



Thus equation of the circle in the new position is

$\{x - (2 + 4\pi)\}^2 + (y - 2)^2 = 2^2$

$\Rightarrow \; x^2 + y^2 - 8\pi x - 4(x + y) + (4\pi + 2)^2 = 0$

23. (b) : Let the equation of the required circle be

$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

Since, the circle is passing through the points (1, 0) and (0, 1). Therefore $1 + 2g + c = 0$ and $1 + 2f + c = 0$

$$\therefore \quad g = -\left(\frac{c+1}{2}\right) \text{ and } f = -\left(\frac{c+1}{2}\right)$$

Let the radius of the circle (i) be r.

$$\therefore \quad r = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{\left(\frac{c+1}{2}\right)^2 + \left(\frac{c+1}{2}\right)^2 - c} = \sqrt{\frac{c^2+1}{2}}$$

Clearly, r will be minimum, when $c = 0$ and the smallest possible radius is $r = \dfrac{1}{\sqrt{2}}$.

$\therefore$ Required equation is $x^2 + y^2 - x - y = 0$

24. (b) : The point $(2a, a + 1)$ will be an interior point of the larger segment of the circle $x^2 + y^2 - 2x - 2y - 8 = 0$. Since, $(2a, a + 1)$ is an interior point and $(2a, a + 1)$ and the centre (1, 1) are on the same side of the chord $x - y + 1 = 0$.

$\therefore \; (2a)^2 + (a + 1)^2 - 2(2a) - 2(a + 1) - 8 < 0$

and $(2a - a - 1 + 1)(1 - 1 + 1) > 0$

$\Rightarrow \; 5a^2 - 4a - 9 < 0$ and $a > 0$

$\Rightarrow \; (5a - 9)(a + 1) < 0$ and $a > 0$

$$\Rightarrow \; -1 < a < \frac{9}{5} \text{ and } a > 0. \text{ Hence } a \in \left(0, \frac{9}{5}\right).$$

25. (b) : Here, the equation of the circle is

$x^2 + y^2 - x + 3y = 0$

$$\therefore \text{ Centre is } \left(\frac{1}{2}, -\frac{3}{2}\right) \text{ and radius} = \sqrt{\frac{5}{2}}.$$

Let the equation of L_1 be $y = mx$. Since intercepts made by the lines are equal. So distances of the lines from the centre are also equal *i.e.*, distance of $mx - y = 0$ and $x + y = 1$ from $\left(\frac{1}{2}, -\frac{3}{2}\right)$ are equal.

Monthly Test Drive-3 CLASS XI ANSWER KEY

1.	(d)	**2.**	(c)	**3.**	(c)	**4.**	(a)
5.	(c)	**6.**	(b)	**7.**	(c, d)	**8.**	(b, c)
9.	(a, b, c)	**10.**	(a, b, c, d)	**11.**	(a, b)	**12.**	(a, b)
13.	(a, b, c, d)	**14.**	(b)	**15.**	(c)	**16.**	(b)
17.	(4950)	**18.**	(785)	**19.**	(6)	**20.**	(7)

$$\therefore \quad \left|\frac{\frac{m}{2}+\frac{3}{2}}{\sqrt{1+m^2}}\right| = \left|\frac{\frac{1}{2}-\frac{3}{2}-1}{\sqrt{1+1}}\right| \Rightarrow \left|\frac{m+3}{\sqrt{1+m^2}}\right| = \left|\frac{-4}{\sqrt{2}}\right|$$

$\Rightarrow \quad (m + 3)^2 = 8(1 + m^2) \Rightarrow 7m^2 - 6m - 1 = 0$

$\Rightarrow \quad (m - 1)(7m + 1) = 0 \Rightarrow m = 1, -1/7$

So, the equations of line L_1 are $y = x$ and $7y = -x$.

26. (a) : The equation of a tangent to $x^2 + y^2 = 1$ is $x \cos\alpha + y\sin\alpha = 1$ ...(i)

The equation of tangent to $x^2 + y^2 = 3$, perpendicular to (i), is $x \sin\alpha - y\cos\alpha = \sqrt{3}$ ...(ii)

Let the coordinate of P be (h, k). Then, $h\cos\alpha + k\sin\alpha = 1$ and $h\sin\alpha - k\cos\alpha = \sqrt{3}$

Eliminating (h, k) from the above two equations, we get $h^2 + k^2 = 4$.

Thus, the required locus is $x^2 + y^2 = 4$, which is a circle of radius 2.

27. (b) : The centre and radius of the circle $x^2 + y^2 = 4$ are C_1 (0, 0) and $r_1 = 2$ respectively.

Also, the centre and radius of the circle $x^2 + y^2 - 6x - 8y = 24$ are C_2 (3, 4) and $r_2 = 7$ respectively.

Distance between the centres of the circles is

$$C_1C_2 = \sqrt{(3-0)^2 + (4-0)^2} = 5 < \text{sum of the radii}$$

Also, C_1C_2 = Difference of the radii.

So, the two circles touch each other internally. Hence, there is only one common tangent.

28. (d) : Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)

Since, circle (i) cuts the circle $x^2 + y^2 - 20x + 4 = 0$ orthogonally.

$\therefore\ 2(-10g + 0 \times f) = c + 4 \ \Rightarrow\ -20g = c + 4$ ...(ii)

Now (i) touch the line $x + 0y - 2 = 0$

$$\therefore \quad \left|\frac{-g+0-2}{\sqrt{1^2+0^2}}\right| = \sqrt{g^2 + f^2 - c}$$

$\Rightarrow \quad (g + 2)^2 = g^2 + f^2 - c \Rightarrow 4g + 4 = f^2 - c$ ...(iii)

Eliminating c from (ii) and (iii), we get

$-16g + 4 = f^2 + 4 \Rightarrow f^2 + 16g = 0.$

Hence, the locus of $(-g, -f)$ is $y^2 - 16x = 0$

29. (c) : The centre and radius of the circle $x^2 + y^2 = 9$ are $C_1(0, 0)$ and $r_1 = 3$

The centre and radius of the circle $x^2 + y^2 - 8x - 6y + n^2 = 0$ are C_2 (4, 3) and $r_2 = \sqrt{25 - n^2}$, $-5 < n < 5$

Given, circles will have exactly two common tangents, if

$|r_1 - r_2| < C_1C_2 < r_1 + r_2$

$\Rightarrow \quad |3 - \sqrt{25 - n^2}\,| < 5 < 3 + \sqrt{25 - n^2}$

Clearly, $|3 - \sqrt{25 - n^2}\,| < 5$ is true for all $n \in (-5, 5)$

Now, $5 < 3 + \sqrt{25 - n^2} \Rightarrow 2 < \sqrt{25 - n^2}$

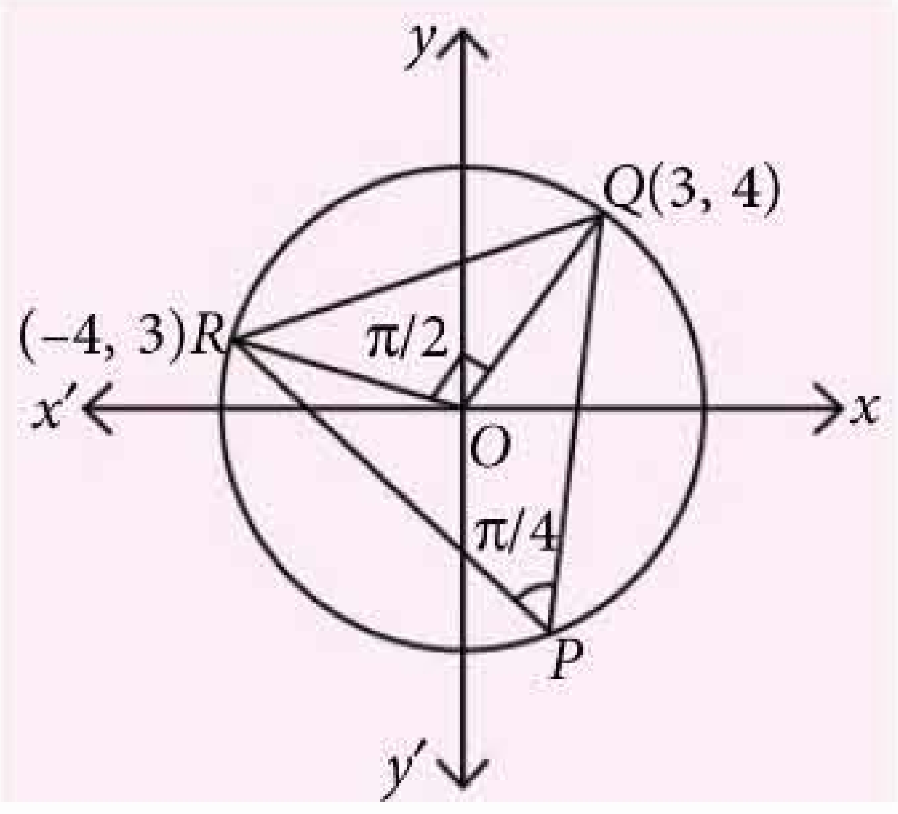
$\Rightarrow \ 4 < 25 - n^2 \Rightarrow n^2 - 21 < 0 \Rightarrow -\sqrt{21} < n < \sqrt{21}$

$\Rightarrow \ n = \pm 4, \pm 3, \pm 2, \pm 1, 0$

Hence, n can take 9 integral values.

30. (c) : Let O be the centre of the circle.

Then, $\angle QOR = 90°$. Hence $\angle QPR = \dfrac{\pi}{4}$.



◆ ◆

1019 Kendriya Vidyalaya students crack IIT-JEE Advanced 2019

The JEE Advanced result was announced on June 14 and 6094 KVS students had appeared for it. JEE (Advanced) is conducted for admission in the Indian Institute of Technology (IITs) located across the country.

As many as 1019 students of Kendriya Vidyalaya Sangathan have cracked the Joint Entrance Examination (JEE) Advanced 2019 conducted by IIT-Roorkee.

Kendriya Vidyalaya Sangathan is an autonomous body under the Union Ministry of Human Resource Development (HRD). The JEE Advanced 2019 was held on May 27, 2019. A total number of 1,61,319 candidates appeared in both papers 1 and 2 in JEE (Advanced) of which, 38705 candidates qualified. Of them 5356 are female candidates. Kartikey Chandresh Gupta of Ballarpur is the top rank holder in Common Rank List in JEE Advanced 2019. He obtained 346 marks out of 372 marks.

As on date, there are 1205 Kendriya Vidyalayas functioning across the country with three being run in the Embassies of Tehran, Kathmandu and Moscow and catering education to over 12.75 lakh students.

This year, the KVS has also delivered its highest ever result in the CBSE class 12 examination with overall pass percent 98.54 and topped the category of all institutions for the second consecutive year.

OLYMPIAD CORNER

1. In quadrilateral $P_1P_2P_3P_4$, suppose that the diagonals intersect at the point $M \neq P_i$ $(i = 1, 2, 3, 4)$. Let $\angle MP_1P_4 = \alpha_1$, $\angle MP_3P_4 = \alpha_2$, $\angle MP_1P_2 = \beta_1$ and $\angle MP_3P_2 = \beta_2$.

   Prove that $\lambda_{13} = \left|\frac{P_1M}{MP_3}\right| = \frac{\cot\alpha_1 \pm \cot\beta_1}{\cot\alpha_2 \pm \cot\beta_2}$,

   where the +(–1) sign holds if the line segment P_1P_3 is located inside (outside) the quadrilateral.

2. Verify that the following three inequalities hold for positive real x, y and z.

   (i) $x(x-y)(x-z) + y(y-z)(y-x) + z(z-x)(z-y) \geq 0$

   (This is known as Schur's Inequality)

   (ii) $x^4 + y^4 + z^4 + xyz(x+y+z) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$

   (iii) $9xyz + 1 \geq 4(xy + yz + xz)$, where $x + y + z = 1$.

3. In a triangle ΔABC, $\angle C = 2\angle B$. P is a point in the interior of ΔABC satisfying that $AP = AC$ and $PB = PC$. Show that AP trisects the angle $\angle A$.

4. Calculate $\int_{-\pi/2}^{\pi/2} \frac{\sin(2N+1)\theta}{\sin\theta}\, d\theta$, where N is a non-negative integer.

5. $P_1, P_2, \ldots, P_{11}$ are eleven distinct points on a line. $P_iP_j \leq 1$ for every pair P_i, P_j. Prove that the sum of all (55) distances $P_i P_j$, $1 \leq i < j \leq 11$ is smaller than 30.

6. Let $A_1, A_2, \ldots, A_8$ be the vertices of a parallelepiped and let O be its centre. Show that $4(OA_1^4 + OA_2^2 + \ldots + OA_8^2) \leq (OA_1 + OA_2 + \ldots + OA_8)^2$.

7. Let x, y be positive integers with $y > 3$ and $x^2 + y^4 = 2[(x-6)^2 + (y+1)^2]$. Prove that $x^2 + y^4 = 1994$.

SOLUTIONS

1. Assume M to be any point of the line P_1P_3 except P_1, P_3, and left P_2P_4 be any other line through M with P_2, P_4 on opposite sides of M. Let $\omega = \angle P_1MP_4$. Then $\sin\angle P_1P_4M = \sin(\pi - (\alpha_1 + \omega)) = \sin(\alpha_1 + \omega)$, while $\sin\angle P_3P_4M = \sin(\omega \pm \alpha_2)$ where '+' corresponds to the subcase where M is outside the segment P_1P_3, while '–' corresponds to where M is between P_1 and P_3. From the sine rule applied to triangles MP_1P_4 and MP_3P_4, we have

$$P_1M\frac{\sin\alpha_1}{\sin(\alpha_1+\omega)} = MP_4 = MP_3\frac{\sin\alpha_2}{\sin(\omega\pm\alpha_2)}$$

Hence, $\frac{P_1M}{MP_3} = \frac{\sin\alpha_2\sin(\alpha_1+\omega)}{\sin(\omega\pm\alpha_2)\sin\alpha_1}$

$$= \frac{\sin\alpha_2\sin\alpha_1\cos\omega + \sin\alpha_2\cos\alpha_1\sin\omega}{\sin\omega\cos\alpha_2\sin\alpha_1 \pm \cos\omega\sin\alpha_2\sin\alpha_1}$$

$$= \frac{\cot\omega + \cot\alpha_1}{\cot\alpha_2 \pm \cot\omega} \qquad \text{...(i)}$$

(where we divided both the numerator and the denominator by the product of three sines to get the last equality).

Similarly, from triangles MP_1P_2 and MP_3P_2, we have

$$\frac{P_1M}{MP_3} = \frac{\sin\beta_2\sin(\omega-\beta_1)}{\sin\beta_1\sin(\omega\mp\beta_2)} = \frac{\cot\beta_1 + \cot\omega}{\cot\beta_2 \mp \cot\omega} \qquad \text{...(ii)}$$

The result follows by combining (i) and (ii) using the following property of proportions :

if $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then $\frac{a}{b} = \frac{(c+e)}{(d+f)}$.

2. We shall show that all three inequalities are equivalent to showing that

$$(x + y - z)(x - y)^2 + (y + z - x)(y - z)^2 + (z + x - y)(z - x)^2 \geq 0.$$

And we shall then show that this inequality holds for all positive reals x, y and z.

Let $p = (x + y - z)(x - y)^2 + (y + z - x)(y - z)^2 + (z + x - y)(z - x)^2$.

Expanding and rearranging terms, we find that

$$\begin{aligned} &2x(x - y)(x - z) + 2y(y - z)(y - x) + 2z(z - x)(z - y) \\ &= 2x^3 + 2y^3 + 2z^3 + 6xyz - 2(x^2y + xy^2 + x^2z + xz^2 + y^2z + yz^2) \\ &= (x^3 + y^3 + 2xyz - x^2y - x^2z - y^2x - y^2z) \\ &\quad + (y^3 + z^3 + 2xyz - y^2x - y^2z - z^2x - z^2y) \\ &\quad + (z^3 + x^3 + 2xy - z^2x - z^2y - x^2y - x^2z) \\ &= (x + y - z)(x - y)^2 + (y + z - x)(y - z)^2 \\ &\quad + (z + x - y)(z - x)^2 = p. \end{aligned}$$

Therefore, we have shown that

$x(x-y)(x-z) + y(y-z)(y-x) + x(z-x)(z-y) = p/2$

Thus (i) holds if and only if $p \geq 0$.

Note that $x^4 + y^4 + z^4 - x^2y^2 - y^2z^2 - z^2x^2$

$$= \frac{1}{2}[(x^2 - y^2)^2 + (y^2 - z^2)^2 + (z^2 - x^2)^2]$$

Furthermore, $x^2y^2 + y^2z^2 + z^2x^2 - xyz(x + y + z)$

$$= \frac{1}{2}[x^2(y - z)^2 + y^2(x - z)^2 + z^2(x - y)^2].$$

Subtracting the second equation from the first, we have

$$x^4 + y^4 + z^4 + xyz(x + y + z) - 2(x^2y^2 + y^2z^2 + z^2x^2)$$

$$= \frac{1}{2}[(x^2 - y^2)^2 + (y^2 - z^2)^2 + (z^2 - x^2)^2 - x^2(y - z)^2 - y^2(x - z)^2 - z^2(x - y)^2]$$

$$= \frac{1}{2}\{(x - y)^2 [(x + y)^2 - z^2] + (y - z)^2 [(y + z)^2 - z^2] + (z - x)^2[(z + x)^2 - y^2]\}$$

$$= \frac{x+y+}{2}.[(x - y)^2 (x + y - z) + (y - z)^2 (y + z - x) + (z - x)^2 (z + x - y)]$$

$$= \frac{p(x+y+z)}{2}.$$

Since x, y and z are positive, $x + y + z$ is also positive, and it follows that (ii) holds if and only if $p \geq 0$.

Suppose that $x + y + z = a$. Then if we replace x by x/a, y by y/a and z by z/a, then the inequality will remain the same. Thus, there is no loss in generality in assuming that $x + y + z = 1$, for we can always divide each term by some constant to ensure that the sum of the terms becomes 1.

Thus, letting $x + y + z = 1$, we have

$$\frac{p}{2} = \frac{1}{2}[(x + y - z)(x - y)^2 + (y + z - x)(y - z)^2 + (z + x - y)(z - x)^2]$$

$$= (x^3 + y^3 + z^3) + 3xyz - (x^2y + x^2z + y^2x + y^2z + z^2x + z^2y)$$

$$= (x^3 + y^3 + z^3) + 3xyz - (x^2 + y^2 + z^2)(x + y + z) + (x^3 + y^3 + z^3)$$

$$= 2(x^3 + y^3 + z^3) + 3xyz - (x^2 + y^2 + z^2)$$

$$= 2(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) + 6xyz + 3xyz - (x^2 + y^2 + z^2)$$

$$= x^2 + y^2 + z^2 - 2(xy + yz + zx) + 9xyz$$

$$= (x + y + z)^2 - 4(xy + yz + zx) + 9xyz$$

$$= 1 - 4(xy + yz + zx) + 9xyz.$$

Therefore, (iii) holds if and only if $p \geq 0$. Hence, we have shown that all three inequalities are equivalent to $p \geq 0$. Thus, if we can show that this inequality holds, then (i), (ii) and (iii) all hold.

To show that $p \geq 0$, we separate the problem into two cases. If x, y and z are the sides of a triangle, then $x + y - z$, $x + z - y$ and $y + z - x$ are all positive and thus it immediately follows that $p \geq 0$. Otherwise, the three sides do not form the sides of a triangle, and so, if we assume without loss of generality, that $x \leq y \leq z$, then $x + y - z \leq 0$. And we have

$$p = (x + y - z)(x - y)^2 + (y + z - x)(y - z)^2 + (z + x - y)(z - x)^2$$

$$= (x + y - z)[(x - z) + (z - y)]^2 + (y + z - x)(y - z)^2 + (z + x - y)(z - x)^2$$

$$=(x - z)^2 (x + y - z) + (z - y)^2 (x + y - z) + 2(x - z)(z - y)(x + y - z) + (y + z - x)(y - z)^2 + (x + z - y)(z - x)^2$$

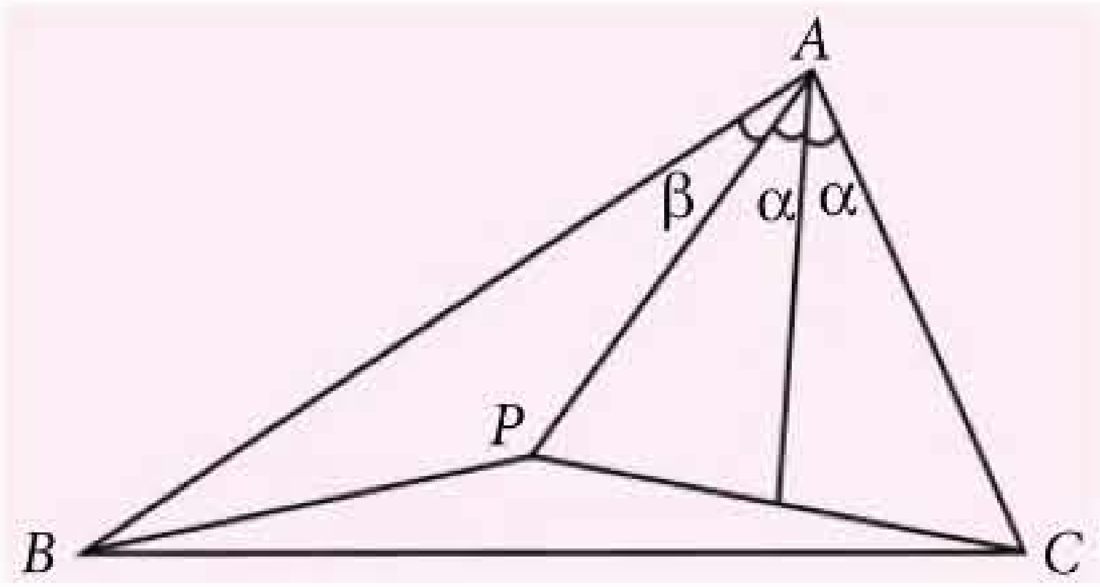
$$= (x - z)^2 (2x) + (y - z)^2 (2y) + 2(z - x)(z - y)(z - x - y).$$

And this last expression is non-negative, since $x \geq 0$, $y \geq 0$, $z - x \geq 0$, $z - y \geq 0$ and $z - x - y \geq 0$.

Therefore, $p \geq 0$ and we have proven that all three inequalities hold.

3. Let $\angle PAC$ and $\angle BAP$ be 2α and β respectively. Then, since $\angle C = 2\angle B$, we deduce from

$A + B + C = 180°$ that $2\alpha + \beta + 3B = 180°$. ...(i)



The angles at the base of the isosceles triangle PAC are each $90° - \alpha$. Also ΔBPC is isosceles, having base angles $C - (90° - \alpha) = 2B + \alpha - 90°$.

and so $\angle BPA = 180° - (\angle PBA + \angle BAP)$

$= 180° - [B - (2B + \alpha - 90°) + 180° - 2\alpha - 4B]$

$= 4B + 3\alpha - 90°$

As usual, let a, b and c denote the lengths of the sides BC, AC and AB. By the law of cosines, applied to ΔBPA, where $\overline{PA} = b$ and $\overline{PB} = \overline{PC} = 2b \sin \alpha$.

$c^2 = b^2 + (2b \sin \alpha)^2 - 2{\cdot}b{\cdot}2b \sin \alpha{\cdot}\cos(4B + 3\alpha - 90°)$,

so that $c^2 = b^2[1 + 4\sin^2\alpha - 4\sin \alpha \sin(4B + 3\alpha)]$ …(ii)

We now use the fact that $\angle C = 2\angle B$ is equivalent to the condition $c^2 = b(b + a)$.

Since $a = 2{\cdot}\overline{PC}{\cdot}\cos(2B + \alpha - 90°)$

$= 4b \sin \alpha \sin(2B + \alpha)$, we have

$c^2 = b^2[1 + 4\sin \alpha \sin(2B + \alpha)]$ ...(iii)

Therefore, from (ii) and (iii), we get

$b^2[1 + 4\sin^2\alpha - 4\sin \alpha \sin(4B + 3\alpha)]$

$= b^2[1 + 4\sin \alpha \sin(2B + \alpha)]$, which simplifies to

$\sin \alpha - \sin(4B + 3\alpha) = \sin(2B + \alpha)$.

Since $\sin\alpha - \sin(4B + 3\alpha) = -2\cos(2B + 2\alpha) \sin(2B + \alpha)$, this equation may be rewritten as

$\sin(2B + \alpha)\cdot[1 + 2\cos(2B + 2\alpha)] = 0$.

Since, from (i), $2B + \alpha < 180°$, we must have $1 + 2\cos(2B + 2\alpha) = 0$, giving $\cos(2B + 2\alpha) = -1/2$; that is,

$2B + 2\alpha = 120°$ ...(iv)

Since, again from (i), $2B + 2\alpha < 180°$.

Finally, we may eliminate B between (i) and (iv) to obtain $\alpha = \beta$.

4. Lef $f(N)$ denote $\dfrac{\sin(2N+1)\theta}{\sin\theta}$. Then

$$f(N+1) - f(N) = \frac{\sin(2N+3)\theta - \sin(2N+1)\theta}{\sin\theta}.$$

Now, $\sin(2N + 3)\theta - \sin(2N + 1)\theta$

$= \sin(2N+1)\theta\cos2\theta + \cos(2N+1)\theta\sin2\theta - \sin(2N+1)\theta$

$= \sin(2N + 1)\theta(\cos2\theta - 1) + \cos(2N + 1)\theta\sin2\theta$

$= -2\sin(2N + 1)\theta\sin2\theta + 2\cos(2N + 1)\theta\sin\theta\cos\theta$

$= 2\sin\theta\,[\cos(2N + 1)\theta\cos\theta - \sin(2N + 1)\theta\sin\theta]$

$= 2\sin\theta\cos(2N + 2)\theta$.

These two equations imply $f(N + 1) - f(N) = 2\cos(2N + 2)\theta$.

So, $\displaystyle\int_{-\pi/2}^{\pi/2} f(N+1)d\theta - \int_{-\pi/2}^{\pi/2} f(N)d\theta$

$\displaystyle= 2\int_{-\pi/2}^{\pi/2} \cos(2N+2)\theta\, d\theta = 0.$

$\displaystyle\therefore\ f(N) = f(0) = \int_{-\pi/2}^{\pi/2} \frac{\sin\theta}{\sin\theta} d\theta = \pi.$

5. Without loss of generality we suppose that $P_1, P_2, ..., P_{11}$ are adjacent.

Now, if $1 \le i < j \le 11$, then $P_iP_j = P_jP_1 - P_iP_1$.

So $\displaystyle\sum_{1\le i<j\le 11} P_iP_j = \sum_{1\le i<j\le 11} \left(P_jP_1 - P_iP_1\right)$

$= 10P_{11}P_1 + 9P_{10}P_1 - P_{10}P_1 + 8P_9P_1 - 2P_9P_1 + 7P_8P_1 - 3P_8P_1 + 6P_7P_1 - 4P_7P_1 + 5P_6P_1 - 5P_6P_1 + 4P_5P_1 - 6P_5P_1 + 3P_4P_1 - 7P_4P_1 + 2P_3P_1 - 8P_3P_1 + P_2P_1 - 9P_2P_1$

$= 10P_{11}P_1 + 8P_{10}P_1 - 6P_9P_1 + 4P_8P_1 + 2P_7P_1 - 2P_5P_1 - 4P_4P_1 - 6P_3P_1 - 8P_2P_1$

$= 10P_{11}P_1 + 8(P_{10}P_1 - P_2P_1) + 6(P_9P_1 - P_3P_1) + 4(P_8P_1 - P_4P_1) + (2P_7P_1 - P_5P_1)$

$= 10P_{11}P_1 + 8P_{10}P_2 + 6P_9P_3 + 4P_8P_4 + 2P_7P_5$

$< 10\cdot1 + 8\cdot1 + 6\cdot1 + 4\cdot1 + 2\cdot1 = 30$

6. Let one of the vertices be the origin and let the vectors $B + C$, $C + A$, $A + B$ denote the three coterminal edges emanating from this origin. Then the vectors to the remaining four vertices are $S + A$, $S + B$, $S + C$ and $2S$, where $S = A + B + C$ and which is also the vector to the centre. The inequality now becomes

$2(S^2 + A^2 + B^2 + C^2) \le (|S| + |A| + |B| + |C|)^2$,

or $S^2 + A^2 + B^2 + C^2 \le 2|S|\,\{|A| + |B| + |C|\} + 2\{|B|\,|C| + |C|\,|A| + |A|\,|B|\}$

Since $S^2 = A^2 + B^2 + C^2 + 2B\cdot C + 2C\cdot A + 2A\cdot B$, the inequality now becomes

$S^2 - B\cdot C - C\cdot A - A\cdot B \le |S|\{|A| + |B| + |C|\} + \{|B|\,|C| + |C|\,|A| + |A|\,|B|\}$.

Clearly, $S^2 \le |S|\{|A| + |B| + |C|\}$

and $(-B\cdot C - C\cdot A - A\cdot B) \le |B|\,|C|\,|A| + |A|\,|B|$.

There is equality if and only if the parallelepiped is degenerate.

7. Rewriting the given equation we get

$x^2 - 24x - y^4 + 2y^2 + 4y + 74 = 0$ ...(i)

Now (i) has integer solutions only if the discriminant $4(y^4 - 2y^2 - 4y + 70)$ is a perfect square. It is easy to prove that for $y \ge 4$.

$(y^2 - 2)^2 < y^4 - 2y^2 - 4y + 70 < (y^2 + 1)^2$. (ii)

[Indeed (ii) $\Leftrightarrow y^2 - 2y + 33 > 0$ and $4y(y + 1) > 69$. The first inequality is true. Since $y \ge 4$, $4y(y + 1) \ge 4\cdot4\cdot5 = 80 > 69$]. The only perfect squares between $(y^2 - 2)^2$ and $(y^2 + 1)^2$ are $(y^2 - 1)^2$ and $(y^2)^2$.

Now $(y^2 - 1)^2 = y^4 - 2y^2 - 4y + 70 \Leftrightarrow y = \dfrac{69}{4} \notin Z$,

and $y^4 - 2y^2 - 4y + 70 = y^4 \Leftrightarrow y^2 + 2y - 35 = 0 \Leftrightarrow y = 5$ or $y = -7$. Thus, $y = 5$.

Now (i) gives $x = 37$ and

$x^2 + y^4 = 37^2 + 5^4 = 1369 + 625 = 1994$.

The result is also true for $y = 1$ and $y = 2$ as well, but fails for $y = 3$ with $x = 1$.

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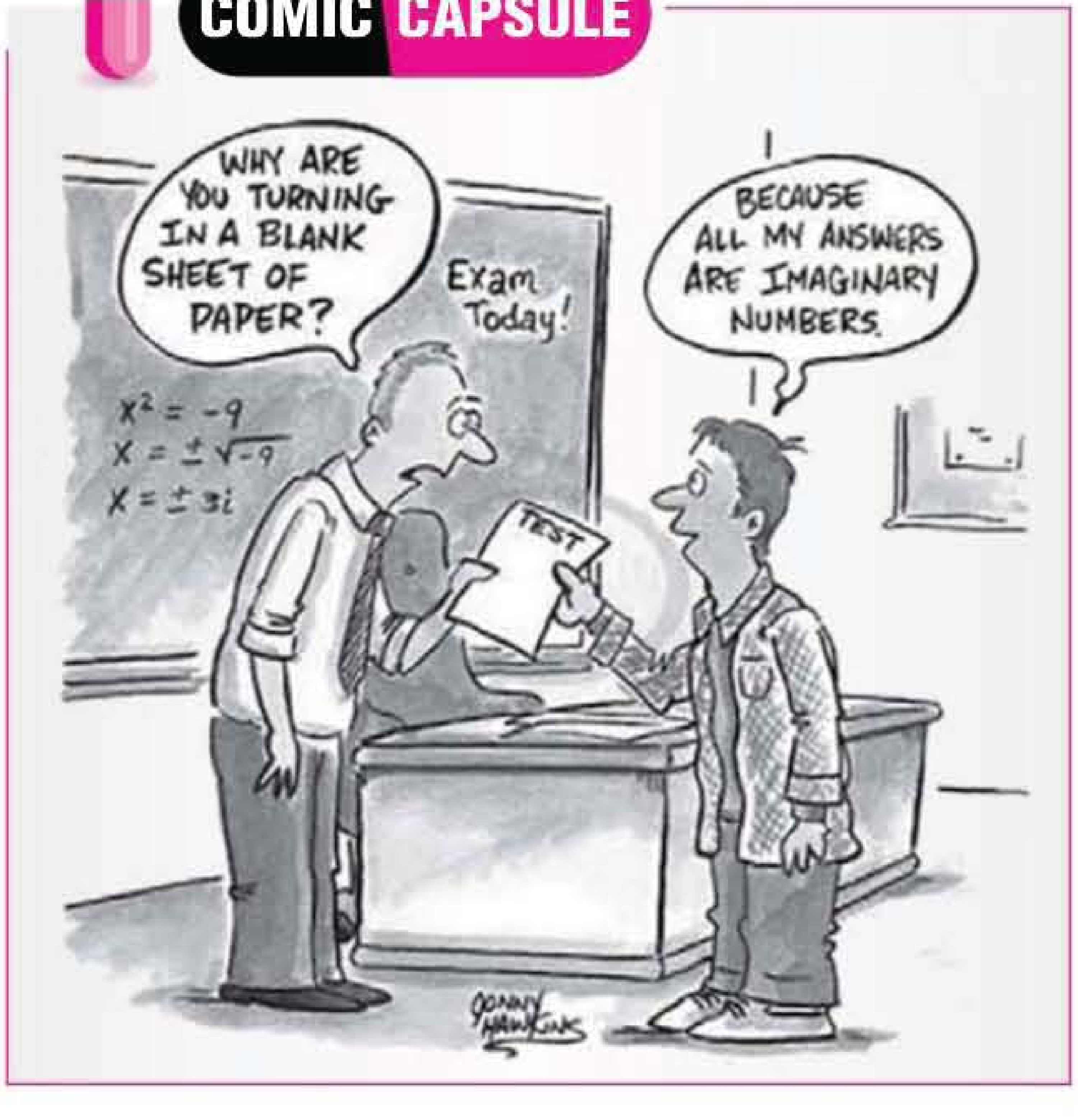
COMIC CAPSULE


WHY ARE YOU TURNING IN A BLANK SHEET OF PAPER?
Exam Today!
$x^2 = -9$
$x = \pm\sqrt{-9}$
$x = \pm 3i$
TEST
BECAUSE ALL MY ANSWERS ARE IMAGINARY NUMBERS.
JONNY HAWKINS

Challenging PROBLEMS ON Limits

1. $\lim_{x\to 0^+}\left(x^{(x^x)}-x^x\right)=$

(a) -1 (b) 0 (c) 1 (d) 2

2. $\lim_{x\to 0}\left(\frac{2}{x^3}(\sin^{-1}x-\tan^{-1}x)\right)^{2/x^2}=$

(a) $\sqrt{e}$ (b) $\frac{1}{\sqrt{e}}$ (c) e (d) $\frac{1}{e}$

3. For $n \in N$, let x_n be defined as $\left(1+\frac{1}{n}\right)^{n+x_n}=e$, then $\lim_{n\to\infty} x_n =$

(a) 0 (b) 1 (c) $1/2$ (d) 2

4. $\lim_{n\to\infty}\left(\frac{{}^{3n}C_n}{{}^{2n}C_n}\right)^{1/n}=$

(a) $\frac{27}{16}$ (b) $\frac{27}{17}$ (c) $\frac{25}{16}$ (d) $\frac{25}{17}$

5. Let $a_1, a_2, \ldots., a_n$ be sequence of real numbers with $a_{n+1}=a_n+\sqrt{1+a_n^2}$, $a_1 = 0$, then $\lim_{n\to\infty}\frac{a_n}{2^{n-1}}=$

(a) $2/\pi$ (b) $\pi/2$ (c) $\pi/4$ (d) $4/\pi$

6. If $\lim_{n\to\infty}\frac{4^n}{(4-\sqrt{3}+2\sin\theta)^{n+2}}=$ finite quantity$(\neq 0)$, where $\theta\in(0, \pi/2)$, then $\tan\theta =$

(a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{\sqrt{3}}{2}$ (d) 1

7. Let $f: R\to(0, \infty)$ be such that $f(x)+\frac{e^{x+x^2}}{f(x)}<e^x+e^{x^2}$, $\forall\ x > 0$, then $\lim_{x\to 1} f(x)=$

(a) e (b) 1 (c) e^2 (d) $2e$

8. $\lim_{x\to\pi}\frac{2^{\cot x}+3^{\cot x}-5^{1+\cot x}+2}{(2^{\cot x})^2+(9^{\cot x})^{1/2}-5^{\cot x}+1}$

(a) equals 2 (b) equals 5

(c) equals 0 (d) does not exist

9. Let $<a_n>$, $<b_n>$, $<c_n>$ be sequences such that

(i) $a_n + b_n + c_n = 2n + 1$ (ii) $a_nb_nc_n = -1$

(iii) $a_nb_n + b_nc_n + c_na_n = 2n - 1$ (iv) $a_n > b_n > c_n$,

then $\lim_{n\to\infty}\frac{a_n}{n}=$

(a) 2 (b) $1/2$ (c) 1 (d) -1

10. If $S_n=\sum_{r=1}^{n}\tan^{-1}\left[\frac{2(2r-1)}{4+r^2(r-1)^2}\right]$, then

$\lim_{n\to\infty}\sum_{r=2}^{n}\cot(S_{r-1})-\cot(S_r)=$

(a) 2 (b) $1/2$ (c) $4/\pi$ (d) $\pi/4$

11. $\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^n}{\left(1-\frac{1}{n}\right)^n}-e^2\right)n^2=$

(a) $\frac{2e^2}{3}$ (b) $\frac{3e^2}{4}$ (c) $\frac{4e^2}{5}$ (d) $\frac{5e^2}{6}$

12. $\lim_{n\to\infty}(1+n+n\cos n)^{\frac{1}{2n+n\sin n}}$

(a) equals 0 (b) equals 1

(c) equals 1/2 (d) does not exist

13. $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\sqrt{1+\frac{k}{n^2}}-1\right)$

(a) equals 1 (b) equals 1/2

(c) equals 1/4 (d) does not exist

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14. Let sequences $<a_n>$ and $<b_n>$ be defined by setting $a_1 = 3$, $b_1 = 2$, $a_{n+1} = a_n + 2b_n$, $b_{n+1} = a_n + b_n$ and $c_n = \frac{a_n}{b_n}, n \in N$. Then $\lim\limits_{n\to\infty} c_n =$

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

15. $\lim\limits_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+....+\frac{1}{\sqrt{n^2+n+1}}\right)=$

(a) 1 (b) e (c) $\pi/4$ (d) $e/2$

16. Let $a_n = 3-\sum\limits_{k=1}^{n}\frac{1}{k(k+1)(k+1)!}, n \in N$, then $\lim\limits_{n\to\infty} a_n =$

(a) e^2 (b) $\sqrt{e}$ (c) $e/2$ (d) e

17. Let f be differentiable on $(0, \infty)$, $a > 0$ and $\lim\limits_{x\to\infty}(af(x)+2\sqrt{x}f'(x))=\lambda$, then $\lim\limits_{x\to\infty} f(x)=$

(a) λa (b) $\frac{\lambda}{a}$ (c) $\frac{\sqrt{\lambda}}{a}$ (d) $\frac{\lambda}{\sqrt{a}}$

18. If $f(x)=\lim\limits_{m\to\infty}\frac{\sin(\pi x^2)+(x+2)^m\cdot\tan x}{x^2+(x+2)^m}$, then $\lim\limits_{x\to -1} f(x)$

(a) equals 0 (b) equals –tan 1

(c) equals tan 1 (d) does not exist

SOLUTIONS

1. (a) : Consider $\lim\limits_{x\to 0^+} x^x = e^{\lim\limits_{x\to 0^+} x\log x}$

$= e^{\lim\limits_{x\to 0^+}\frac{\log x}{1/x}} = e^{\lim\limits_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)}$ [By L.H. Rule]

$= e^{-0} = 1$

Hence, given limit = 0 – 1 = –1

2. (b) : Simplify using series expansions of $\sin^{-1}x$ and $\tan^{-1}x$.

3. (c) : Simplifying for x_n, we have

$$x_n = \frac{1}{\log\left(1+\frac{1}{n}\right)} - n$$

So, $\lim\limits_{n\to\infty} x_n = \lim\limits_{t\to 1}\left(\frac{1}{\log t}-\frac{1}{t-1}\right)$, where $1+\frac{1}{n}=t$

$= \lim\limits_{t\to 1}\frac{1/t^2}{\frac{1}{t^2}+\frac{1}{t}}$ [By L.H. Rule] $= \frac{1}{2}$

4. (a) : Consider

$$\frac{^{3n}C_n}{^{2n}C_n} = \frac{(3n)!(n)!}{(2n)!(2n)!} \cong \frac{(3n)^{3n}\cdot(n)^n}{(2n)^{4n}} = \left(\frac{27}{16}\right)^n$$

So, $\lim\limits_{n\to\infty}\left(\left(\frac{27}{16}\right)^n\right)^{1/n} = \frac{27}{16}$

5. (d) : Let $a_n = \cot(\alpha_n)$, then

$a_{n+1} = \cot(\alpha_n) + \mathrm{cosec}(\alpha_n) \Rightarrow a_{n+1} = \cot\left(\frac{\alpha_n}{2}\right)$

Putting $n = 1$, we have, $a_1 = \cot(\alpha_1) = 1 \Rightarrow \alpha_1 = \frac{\pi}{4}$

Now, $a_2 = \cot\left(\frac{\alpha_1}{2}\right) = \cot\left(\frac{\pi}{8}\right)$ etc $\therefore a_n = \cot\left(\frac{\pi}{4\cdot 2^{n-1}}\right)$

So, required limit $= \lim\limits_{n\to\infty}\frac{\cot\left(\frac{\pi}{4\cdot 2^{n-1}}\right)}{2^{n-1}} = \frac{4}{\pi}$

6. (a) : Let

$$L = \lim_{n\to\infty}\left(\frac{4}{4-\sqrt{3}+2\sin\theta}\right)^n\cdot\frac{1}{(4-\sqrt{3}+2\sin\theta)^2}$$

For $\theta \in (0, 60°)$, $\sin\theta \in \left(0, \frac{\sqrt{3}}{2}\right)$

$\therefore$ L does not exist in this case.

For $\theta \in (60°, 90°)$, $\sin\theta \in \left(\frac{\sqrt{3}}{2}, 1\right) \Rightarrow L = 0$

Hence, for L to exist and $\neq 0$, $\theta = 60°$

So, $\tan\theta = \sqrt{3}$

7. (a) : Rewriting the given equation

$(f(x)-e^x)(f(x)-e^{x^2}) \le 0$

So, for $x \in (0,1)$, $e^{x^2} \le f(x) \le e^x$ and for $x \in (1,\infty)$

$e^x \le f(x) \le e^{x^2}$.

In any case $\lim\limits_{x\to 1} f(x) = e$ [By Sandwich Rule]

8. (d) : R.H.L. $= \lim\limits_{x\to\pi^+}\frac{\left(\frac{2}{5}\right)^{\cot x}+\left(\frac{3}{5}\right)^{\cot x}-5+\frac{2}{5^{\cot x}}}{\left(\frac{4}{5}\right)^{\cot x}+\left(\frac{3}{5}\right)^{\cot x}-1+\frac{1}{5^{\cot x}}}$

$= \frac{-5}{-1} = 5$

and L.H.L. $= \lim\limits_{x\to\pi^-}\frac{2^{\cot x}+3^{\cot x}-5\cdot 5^{\cot x}+2}{4^{\cot x}+3^{\cot x}-5^{\cot x}+1} = \frac{2}{1} = 2$

As R.H.L. ≠ L.H.L., though both limits are finite. Hence, required limit does not exist.

9. (a) : Let a_n, b_n, c_n are roots of the cubic equation $y^3 - (2n + 1)y^2 + (2n - 1)y + 1 = 0$

Solving, the above equation we get the roots as

$$1, n \pm \sqrt{n^2+1}$$

$$\therefore \quad a_n = n + \sqrt{n^2+1}$$

So, $\lim\limits_{n\to\infty} \dfrac{a_n}{n} = \lim\limits_{n\to\infty}\left(1+\sqrt{1+\dfrac{1}{n^2}}\right) = 2.$

10. (a) : Here, $S_n = \sum\limits_{r=1}^{n}\left[\tan^{-1}\left(\dfrac{r^2}{2}\right) - \tan^{-1}\dfrac{(r-1)^2}{2}\right]$

i.e., $S_n = \tan^{-1}\left(\dfrac{n^2}{2}\right)$

Hence, required limit $= \lim\limits_{n\to\infty}\sum\limits_{r=2}^{n}\left(\dfrac{2}{(r-1)^2} - \dfrac{2}{r^2}\right)$

$$= \lim_{n\to\infty}\left(2 - \frac{2}{n^2}\right) = 2$$

11. (a) : Putting $\dfrac{1}{n} = t$ the limit expression simplifies to

$$e^2 \lim_{t\to 0}\frac{\log(1+t) - \log(1-t) - 2t}{t^3} = e^2 \cdot \left(\frac{2}{3}\right) \text{ [By L.H. Rule]}$$

12. (b) : We have

$$1 < (1 + n(1+\cos n))^{\frac{1}{2n+n\sin n}} < (1+2n)^{\frac{1}{2n+n\sin n}} < (1 + 2n)^{1/n}$$

Hence, by Sandwich theorem at $n \to \infty$, required limit = 1.

13. (c) : Using A.M.- G.M.-H.M. inequation on $x + 1 > 0$, we have

$$1 + \frac{x}{2+x} = \frac{2}{1+\dfrac{1}{1+x}} \le \sqrt{1+x} \le \frac{1+x+1}{2} = 1 + \frac{x}{2}$$

Hence, $\dfrac{x}{2+x} \le \sqrt{1+x} - 1 \le \dfrac{x}{2}$

Putting $x = \dfrac{k}{n^2}, k = 1, 2, ..., n$ and adding all the inequations, we get

$$\sum_{k=1}^{n}\frac{k/n^2}{2+k/n^2} \le \sum_{k=1}^{n}\left(\sqrt{1+\frac{k}{n^2}} - 1\right) \le \sum_{k=1}^{n}\frac{k}{2n^2}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\ge \frac{1}{2n^2+n}\cdot\sum_{k=1}^{n} k = \frac{n(n+1)}{2(2n^2+n)} \qquad = \frac{n(n+1)}{4n^2}$$

Taking $n \to \infty$, using sandwich rule the required limit = 1/4.

14. (b) : Notice that, $c_{n+1} = \dfrac{a_n + 2b_n}{a_n + b_n} = \dfrac{c_n + 2}{c_n + 1}$

$$\text{So, } |c_{n+1} - \sqrt{2}| = \frac{(\sqrt{2}-1)}{c_n+1}|c_n - \sqrt{2}| < (\sqrt{2}-1)|c_n - \sqrt{2}| < \frac{1}{2}|c_n - \sqrt{2}|$$

Iterating, $|c_{n+1} - \sqrt{2}| < \dfrac{1}{2^n}|c_n - \sqrt{2}|$

Hence, $\lim\limits_{n\to\infty}$ gives $c_n = \sqrt{2}$

15. (a) : Note that

$$\frac{n+1}{\sqrt{n^2+n+1}} \le \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + ... + \frac{1}{\sqrt{n^2+n+1}} \le \frac{n+1}{\sqrt{n^2+1}}$$

Now, using sandwich rule, at $n \to \infty$, required limit = 1.

16. (d) : $a_n = 3 - \sum\limits_{k=1}^{n}\left(\dfrac{1}{k} - \dfrac{1}{k+1}\right)\cdot\dfrac{1}{(k+1)!}$

$$= 3 - \sum_{k=1}^{n}\frac{1}{kk!} + \sum_{k=1}^{n}\frac{1}{(k+1)!} + \sum_{k=1}^{n}\frac{1}{(k+1)(k+1)!}$$

$$= \sum_{k=0}^{n+1}\left(\frac{1}{k!} + \frac{1}{(n+1)(n+1)!}\right)$$

Hence, $\lim\limits_{n\to\infty} a_n = e + 0 = e.$

17. (b) : $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\dfrac{e^{a\sqrt{x}}f(x)}{e^{a\sqrt{x}}}$

$$= \lim_{x\to\infty}\frac{e^{a\sqrt{x}}\left[\dfrac{a}{2\sqrt{x}}f(x) + f'(x)\right]}{\dfrac{a}{2\sqrt{x}}\cdot e^{a\sqrt{x}}} \qquad \text{[By L.H. Rule]}$$

$$= \lim_{x\to\infty}\frac{af(x) + 2\sqrt{x}f'(x)}{a} = \frac{\lambda}{a}$$

18. (d) : Taking three cases, according to $x + 2 > 1$, $x + 2 < 1$ or $x + 2 = 1$, we have

$$f(x) = \begin{cases}\tan x, \text{ if } x+2 > 1 \Rightarrow x > -1 \\ \dfrac{\sin(\pi x^2)}{x^2}, \text{ if } 0 < x+2 < 1 \Rightarrow -2 < x < -1\end{cases}$$

Hence, R.H.L. $= \lim\limits_{x\to -1^+} f(x) = -\tan 1$

and L.H.L. $= \lim\limits_{x\to -1^-} f(x) = 0 \quad \therefore$ R.H.L. $\ne$ L.H.L.

Hence required limit does not exist.

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QUANTITATIVE APTITUDE

Useful for Bank PO, Specialist Officers & Clerical Cadre, BCA, MAT, CSAT, CDS and other such examinations

1. A person has five iron rods with lengths 16, 24, 48, 72, 104 cm each. He wants to convert into pieces of equal length from each of five rods. The least number of total pieces, if there is no wastage of material is

(a) 14 (b) 8 (c) 33 (d) 44

2. The average marks of 30 students are 45 but after checking there are two mistakes found. After adjustment if a student got 45 more marks and other student got 15 less mark then what will be the correct average?

(a) 47 (b) 46 (c) 45 (d) 44

3. By selling 8 dozen pencils, a shopkeeper gains the selling price of 1 dozen, what is gain %?

(a) $14\frac{2}{7}\%$ (b) $13\frac{1}{7}\%$ (c) $87\frac{1}{2}\%$ (d) $12\frac{1}{2}\%$

4. A man, a woman, a boy worked for a contractor for the same time. A man is twice efficient as a woman and a woman is thrice efficient as a boy. ₹ 12500 are given to all of them. What is the sum of money got by a man and a boy together?

(a) ₹ 7500 (b) ₹ 3750 (c) ₹ 8750 (d) ₹ 5000

5. What can be said about the expression of $2^{12n} - 6^{4n}$, where n is a positive integer?

(a) Last digit is 4 (b) Last digit is 8

(c) Last two digits are zero

(d) Last digit is 2

6. The value of $\dfrac{(0.92)^3 - (0.1)^3}{(0.92)^2 + .092 + (0.01)}$ is

(a) $0.\overline{82}$ (b) 1.02 (c) 0.82 (d) 0.93

7. Two pipes A and B can fill a cistern in 12 hours and 15 hours respectively. The pipes are opened simultaneously and it is found that due to leakage in the bottom, it took 80 minutes more to fill the cistern. When the cistern is full, in what time will the leakage empty it?

(a) 17 hours (b) 18 hours

(c) 19 hours (d) none of these

8. If a, b, c, d and e are in continued proportion, then $a : e$ equals

(a) $a^4 : b^3$ (b) $a^3 : b^4$ (c) $a^3 : b^3$ (d) $a^4 : b^4$

9. If simple interest on ₹ 2000 increased by ₹ 60 when the rate % increased by 3% p.a., then time equals

(a) 1 year (b) 2 years (c) 3 years (d) 4 years

10. In a rectangular field 12 m wide and 22 m long, we need to pave a path around its borders of uniform width so as to leave an area of 144 sq. m. Width of the path is

(a) 2.5 m (b) 4 m (c) 3 m (d) 2 m

11. $2\frac{1}{2} \text{ of } \frac{3}{4} \times \frac{1}{2} \div \frac{3}{2} + \frac{1}{2} \div \left\{\left[\frac{3}{2}\left(\frac{2}{3} - \frac{1}{2} \text{ of } \frac{2}{3}\right)\right]\right\} = ?$

(a) $2\frac{5}{8}$ (b) $\frac{3}{8}$ (c) $\frac{5}{8}$ (d) $1\frac{5}{8}$

12. In a race of 200 m, A beats B by 31 m and C by 18 m, then by how many metres C will defeat B in a 420 metres race?

(a) 20 m (b) 22 m (c) 25 m (d) 30 m

13. The least value of $(4\sec^2\theta + 9\text{cosec}^2\theta)$ is equal to

(a) 25 (b) 19 (c) 7 (d) 1

14. An increase of 20% on price of banana enable a man to purchase 4 bananas less for ₹ 40. The price of 15 bananas before increase in price was

(a) ₹ 10 (b) ₹ 15 (c) ₹ 20 (d) ₹ 25

15. Select the missing number from the given options.
500, 484, 459, ?, 374

(a) 418 (b) 423 (c) 432 (d) 384

16. The number of arrangements that can be made out from the letters of the word 'SUPERSTITIOUS' is

(a) $\dfrac{12!}{(2!)^3}$ (b) $\dfrac{13!}{3!(2!)^3}$ (c) $\dfrac{13!}{(2!)^2}$ (d) $\dfrac{12!}{(2!)^2}$

17. If $\dfrac{x^2 + y^2 + z^2 - 64}{xy - yz - zx} = -2$ and $x + y = 3z$, then $z =$

(a) 1 (b) 2 (c) 3 (d) 4

18. Reena, Reema and Meera take a car on rent. They use the car for 14 h, 16 h and 22 h respectively. What

amount will be paid by Reema if total rent of car is 2080?
(a) ₹ 1440 (b) ₹ 560 (c) ₹ 880 (d) ₹ 640

19. At what time are the hands together coincide between 2 o'clock and 3 o'clock ?

(a) $10\frac{10}{11}$ mins past 2 (b) $10\frac{9}{11}$ mins past 2
(c) $11\frac{10}{11}$ mins past 2 (d) None of these

20. The height of a bucket is 45 cm. The radii of two circular ends are 28 cm and 7 cm respectively. The volume of the bucket is
(a) 38510 cm^3 (b) 36810 cm^3
(c) 48510 cm^3 (d) 48600 cm^3

21. Three digit numbers are formed by using the digits, 0, 1, 3, 5 and 9. A number is chosen randomly among these numbers, what is the probability that selected number has the same (identical) digits?
(a) 1/50 (b) 1/25 (c) 1/100 (d) 2/25

22. The product of two numbers is 17, then sum of reciprocals of their squares is

(a) $\frac{1}{289}$ (b) $\frac{289}{290}$ (c) $\frac{290}{289}$ (d) $\frac{1}{290}$

23. The calendar 2024 will be same as calendar of year
(a) 2030 (b) 2052
(c) 2035 (d) None of these

24. 25 tonnes of steel cost ₹ 125000. What will be the cost of 270 kg steel?
(a) ₹ 1500 (b) ₹ 1350 (c) ₹ 1450 (d) ₹ 1250

25. If $f(x) = \log_4(\log_5(\log_3(12x - x^2 - 32)))$, then find the domain of $f(x)$.
(a) (–8, –4) (b) (–4, 8) (c) (–7, –5) (d) (5, 7)

26. The population of a town is 126800. It increases by 15% in the first year and decreases 20% in the second year. The population of town at the end of 2 years is
(a) 174984 (b) 116656 (c) 135996 (d) 145820

27. The ratio of the ages of Nishtha and Tina is 3 : 4. After five years the new ratio of their ages will be 4 : 5. What is the age of Nishtha and Tina (in years) respectively?
(a) 20, 15 (b) 15, 20 (c) 12, 16 (d) 21, 28

28. A garrison of 'n' men has enough food to last for 30 days. After 10 days, 50 more joined them. If the food now lasted for 16 days, what is the value of n?
(a) 225 (b) 240 (c) 200 (d) 180

29. In covering a distance 40 km, Ram takes $2\frac{1}{2}$ hrs more than Karan. If the speed of Ram be doubles then his time reduces $1\frac{1}{2}$ hrs less than Karan. The speed of Ram equals
(a) 5 km/h (b) 4 km/h
(c) 3 km/h (d) None of these

30. $\left[\left(\sqrt[5]{x^{-3/5}}\right)^{-5/3}\right]^5 =$

(a) x^5 (b) x^{-5} (c) x (d) x^{-1}

SOLUTIONS

1. **(c) :** Lengths of rods are 16, 24, 48, 72, 104 (cms each)
H.C.F. of (16, 24, 48, 72, 104) = 8

$\therefore$ Least number of total pieces $= \frac{16+24+48+72+104}{8}$
$= 33$

2. **(d) :** Correct observation

$$= \frac{\text{Sum of all observation} - \text{more marks} + \text{less marks}}{\text{Number of observations}}$$

$$= \frac{30 \times 45 - 45 + 15}{30} = \frac{30 \times 44}{30} = 44$$

3. **(a) :** Gain = selling price of 1 dozen
Let the S.P. of 1 dozen be ₹ x.
$\therefore$ S.P. of 8 dozen = ₹ $8x$
Now cost price of 8 dozen pencils
= selling price – profit = $8x - x$ = ₹ $7x$

$\therefore$ Gain % $= \frac{x}{7x} \times 100 = 14\frac{2}{7}\%$

4. **(c) :** From problem
Efficiency of a man : woman : boy
2 : 1
3 : 1
$\therefore$ 2 × 3 : 3 × 1 : 1 × 1 = 6 : 3 : 1

$\therefore$ Share of a man and boy $= \frac{(6+1)}{10} \times 12500$
= ₹ (7 × 1250) = ₹8750

5. **(c) :** $2^{12n} - 6^{4n} = (2^{12})^n - (6^4)^n = (4096)^n - (1296)^n$
$= a^n - b^n$ (where $a = 4096$ and $b = 1296$)
$= (a - b)\,(a^{n-1} + b\,a^{n-2} + b^2\,a^{n-3} + b^3a^{n-4} + \ldots + b^{n-1})$.
$= (a-b)\,(\lambda)$, where $\lambda = (a^{n-1} + b\,a^{n-2} + \ldots + b^{n-1})$
$= (4096 - 1296)\,\lambda = 2800\,\lambda$
Thus, last two digits are zero.

6. **(c)**

7. **(d) :** Part filled by pipe A in 1 hour = 1/12
Part filled by pipe B in 1 hour = 1/15

$\therefore$ Part filled by $(A + B)$ in 1 hour $= \frac{1}{12} + \frac{1}{15} = \frac{3}{20}$

$\therefore$ Time taken by these pipes to fill the cistern
= 20/3 = 6 hours 40 minutes

Due to leakage the time taken = 6 hours 40 minutes + 80 minutes = 8 hours

$\therefore$ Part filled in 1 hour = 1/8

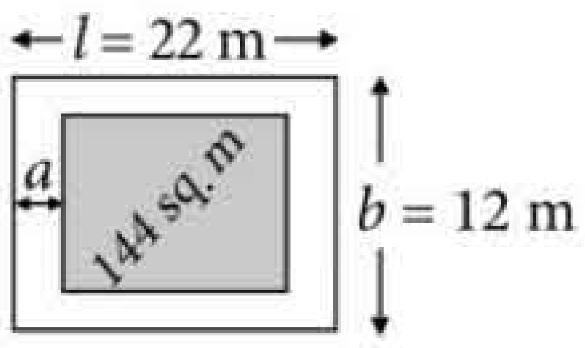
$$\therefore \text{ Part emptied by leakage in 1 hour} = \frac{1}{8} - \frac{3}{20} = \frac{5-6}{40} = \frac{-1}{40}$$

$\therefore$ Leakage can empty the cistern in 40 hours.

8. (d) **9. (a)**

10. (d) : Let the width of the path be 'a' m. Length of field (outer rectangle) = 22 m

Breadth (width) of field (outer rectangle) = 12 m



$\therefore$ Length of inner rectangle = $(22 - 2a)$ m
and Breadth of inner rectangle = $(12 - 2a)$ m
According to problem, $(22 - 2a)(12 - 2a) = 144$
$\Rightarrow (11 - a)(6 - a) = 36 \quad \Rightarrow \quad a^2 - 17a + 30 = 0$
$\Rightarrow (a - 2)(a - 15) = 0$
$\Rightarrow a = 2 \qquad (\because a = 15 \text{ is not possible})$

11. (d)

12. (d) : Distance of race = 200 m
A beats B by 31 m and C by 18 m
$\therefore A : B = 200 : (200 - 31) = 200 : 169$
and $A : C = 200 : (200 - 18) = 200 : 182$

$$\therefore \quad C : B = \frac{C}{B} = \frac{C}{A} \times \frac{A}{B} = \frac{182}{200} \times \frac{200}{169} = 182 : 169$$

When C runs 182 m, B runs 169 m

$$\text{When } C \text{ runs 420 m, then } B \text{ will run } = \frac{169}{182} \times 420 = 390 \text{ m}$$

$\therefore$ C will defeat B by $(420 - 390) = 30$ m.

13. (a)

14. (d) : Let the original price of banana be ₹ x.

$$\therefore \text{ New price is 20\% more than } x = \frac{120 \times x}{100} = ₹\,\frac{6x}{5}$$

Difference in the number of bananas before and after increase of price = 4

$$\therefore \quad \frac{40}{x} - \frac{40 \times 5}{6x} = 4$$

$\Rightarrow$ x (original price of 1 banana) = ₹ (5/3)

$\therefore$ Cost of 15 bananas before increase of price = $(15 \times 5)/3$ = ₹ 25

15. (b)

16. (b) : For the number of arrangements from the word 'SUPERSTITIOUS'
$n = 13$, $S = 3$, $U = 2$, $T = 2$, $I = 2$ and other letters are distinct.

$$\therefore \text{ Required arrangements} = \frac{13!}{3!(2!)^3}$$

17. (d)

18. (d) : Reena's share : Reema's share : Meera's share
= Ratio of time period for which car is used
= 14 : 16 : 22 = 7 : 8 : 11
$\therefore$ Sum of ratio's = 7 + 8 + 11 = 26
Total rent = ₹ 2080

$$\therefore \text{ Amount paid by Reema} = \frac{8 \times 2080}{26} = 8 \times 80 = ₹\,640$$

19. (a) : Here $n = 2$
$\therefore$ The time when hands be together

$$= \left(5 \times n \times \frac{60}{55}\right) \text{ mins past } n \text{ o'clock}$$

$$= 5 \times 2 \times \frac{60}{55} = \frac{120}{11} = 10\frac{10}{11} \text{ mins past 2}$$

20. (c)

21. (b) : Given digits are 0, 1, 3, 5, 9
$\therefore$ Total possible 3-digit numbers = 4 × 5 × 5 (as repetition is allowed) = 100 out of which only 111, 333, 555, 999 (*i.e.*, 4 numbers) having all digits identical.

$$\therefore \text{ Required probability} = \frac{4}{100} = \frac{1}{25}.$$

22. (c)

23. (b) : The year 2024 is a leap year and a leap year in 99% case occurs after 28 years. (Note : some time leap year repeated after 12 years or 40 years)
$\therefore$ 2024 + 28 = 2052.

24. (b) : $\because$ 1 tonne = 1000 kg
$\therefore$ 25 tonnes = 1000 × 25 = 25000 kg
Now, cost of 25000 kg steel = ₹ 125000 [Given]

$$\text{Cost of 1 kg steel} = \frac{125000}{25000} = ₹\,5$$

$\therefore$ Cost of 270 kg steel = ₹ (5 × 270) = ₹ 1350

25. (d)

26. (b) : Here $r_1 = 15\%$, $r_2 = 20\%$ and $P_0 = 126800$
$\therefore$ Population at the end of 2 years

$$= 126800 \times \left(1 + \frac{15}{100}\right)\left(1 - \frac{20}{100}\right) = 116656$$

27. (b)

28. (c) : Here $x = n$, $y = 50$, $a = 30$, $b = 10$,
Sufficient days = 16

$$\therefore \text{ Required no. of days } i.e., \ 16 = \frac{n(30 - 10)}{n + 50}$$

$\Rightarrow (n + 50) \times 16 = 20n \Rightarrow n = 200$

29. (a) **30. (c)**

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The Magic of Vedic Mathematics

Multiplication of Two Digit Multiplicand by Two Digit Multiplier

Step 1 : The right hand digit of multiplicand is multiplied by right hand digit of multiplier.

Step 2 : Apply inside-outside principle (plus carry).

Step 3 : The left hand digit of multiplicand is multiplied by left hand digit of multiplier (plus carry.)

Example : Multiply 56 by 17.

$$\underline{56 \times 17}$$

Step 1 : Right hand digit 6 of multiplicand 56 is multiplied by right hand digit 7 of multiplier 17

6 × 7 = 42; write 2, carry over 4

$$\begin{array}{r}5\overset{\frown}{6 \times 17}\\ \hline 2\end{array}$$

Step 2 : Apply inside-outside principle

[(6 × 1) + (5 × 7)] = 6 + 35 = 41

Plus carry 4;

41 + 4 = 45; write 5, carry over 4

$$\begin{array}{r}\overset{\frown}{56 \times 17}\\ \hline 52\end{array}$$

Step 3 : Left hand digit 5 of multiplicand 56 is multiplied by left hand digit 1 of multiplier 17.

5 × 1 = 05

Plus carry 4;

05 + 4 = 09; write 09

The answer is 952

$$\begin{array}{r}\overset{\frown}{56 \times 1}7\\ \hline 0952\end{array}$$

Multiplication by Two Digit Multiplier

Step 1 : The right most digit of multiplicand is multiplied by right most digit of multiplier.

Step 2 : For each successive pair of digits of multiplicand starting from second right most digit and multiplier apply inside-outside principle (plus carry).

Step 3 : The left most digit of multiplicand is multiplied by left most digit of multiplier (plus carry.)

Example : Multiply 987432 by 56.

$$\underline{987432 \times 56}$$

Step 1 : Right most digit 2 multiplicand is multiplied by right most digit 6 of multiplier

2 × 6 = 12; write 2, carry over 1

$$\begin{array}{r}98743\overset{\frown}{2 \times 56}\\ \hline 2\end{array}$$

Step 2 :

- For Pair 32 and Multiplier 56,
  apply inside-outside principle + (carry 1)
  [(2 × 5) + (3 × 6)] + 1
  = 10 + 18 + 1 = 29;
  write 9, carry over 2

$$\begin{array}{r}9874\overset{\frown}{32 \times 56}\\ \hline 92\end{array}$$

- For pair 43 and multiplier 56
  apply inside-outside principle + (carry 2)
  [(3 × 5) + (4 × 6)] + 2
  = 15 + 24 + 2 = 41;
  write 1, carry over 4

$$\begin{array}{r}987\overset{\frown}{432 \times 56}\\ \hline 192\end{array}$$

- For pair 74 and multiplier 56
  apply inside-outside principle + (carry 4)
  [(4 × 5) + (7 × 6)] + 4
  = 20 + 42 + 4 = 66;
  write 6, carry over 6

$$\begin{array}{r}98\overset{\frown}{7432 \times 56}\\ \hline 6192\end{array}$$

- For pair 87 and multiplier 56
  Apply inside-outside principle + (carry 6)
  [(7 × 5) + (8 × 6)] + 6 = 35 + 48 + 6 = 89;
  write 9, carry over 8

$$\begin{array}{r}9\overset{\frown}{87432 \times 56}\\ \hline 96192\end{array}$$

- For pair 98 and multiplier 56
  apply inside-outside principle + (carry 8)
  [(8 × 5) + (9 × 6)] + 8
  = 40 + 54 + 8 = 102;
  write 2, carry over 10

$$\begin{array}{r}\overset{\frown}{987432 \times 56}\\ \hline 296192\end{array}$$

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Step 3 : The left most digit 9 of multiplicand is multiplied by left most digit 5 of multiplier + (carry 10)
(9 × 5) + 10 = 45 + 10 = 55;
write 55

987432 × 56
55296192

The answer is 55296192

Multiplication by Three Digit Multiplier

Step 1 : The right most digit of multiplicand is multiplied by right most digit of multiplier.

Step 2 : For the right most pair of multiplicand and right most pair of multiplier apply inside-outside principle (plus carry).

Step 3 : For each successive triplet of multiplicand and multiplier apply inside-outside principle (plus carry).

Step 4 : For the left most pair of multiplicand and left most pair of multiplicand apply inside-outside principle (plus carry).

Step 5 : The left most digit of multiplicand is multiplied by left most digit of multiplier (plus carry.)

Example : Multiply 9870321 by 654.

9870321 × 654

Step 1 : Multiply 1 by 4;
1 × 4 = 4;
write 4

9870321 × 654
4

Step 2 : For pair 21 and pair 54 apply inside-outside principle
[(1 × 5) + (2 × 4)] = 5 + 8 = 13;
write 3, carry over 1

9870321 × 654
34

Step 3 :

- For triplet (321) and multiplier (654)
  apply inside-outside principle + (carry 1)
  [(1 × 6) + (2 × 5) + (3 × 4)] + 1
  = 6 + 10 + 12 + 1 = 29;
  Write 9, carry over 2

  9870321 × 654
  934

- For triplet (032) and multiplier (654)
  Apply inside-outside principle + (carry 2)
  [(2 × 6) + (3 × 5) + (0 × 4)] + 2
  = 12 + 15 + 0 + 2 = 29;
  write 9, carry over 2

  9870321 × 654
  9934

- For triplet (703) and multiplier (654)
  apply inside-outside principle + (carry 2)
  [(3 × 6) + (0 × 5) + (7 × 4)] + 2
  = 18 + 0 + 28 + 2 = 48;
  write 8, carry over 4

  9870321 × 654
  89934

- For triplet (870) and multiplier (654)
  apply inside-outside principle + (carry 4)
  [(0 × 6) + (7 × 5) + (8 × 4)] + 4
  = 0 + 35 + 32 + 4 = 71;
  write 1, carry over 7

  9870321 × 654
  189934

- For triplet (987) and multiplier (654)
  apply inside-outside principle + (carry 7)
  [(7 × 6) + (8 × 5) + (9 × 4)] + 7
  = 42 + 40 + 36 + 7 = 125;
  write 5, carry over 12

  9870321 × 654
  5189934

Step 4 : For pair (98) and pair (65)
apply inside-outside principle + (carry 12)
[(8 × 6) + (9 × 5)] + 12
= 48 + 45 + 12 = 105;
write 5, carry over 10.

9870321 × 654
55189934

Step 5 : Multiply 9 by 6 + (carry 10)
(9 × 6) + 10 = 54 + 10 = 64;
write 64

9870321 × 654
6455189934

The answer is 6455189934

SAMURAI SUDOKU

ANSWER - JULY 2019

3	9	4	5	8	1	2	6	7				9	3	4	5	7	8	6	2	1
7	8	6	3	2	9	4	1	5				6	7	1	9	2	4	3	8	5
5	2	1	4	6	7	8	3	9				5	2	8	1	6	3	7	9	4
4	1	5	6	9	3	7	2	8				7	9	5	4	8	1	2	6	3
2	3	9	8	7	4	6	5	1				1	4	2	7	3	6	8	5	9
6	7	8	1	5	2	3	9	4				8	6	3	2	9	5	4	1	7
1	5	2	7	3	8	9	4	6	2	8	1	3	5	7	6	1	2	9	4	8
9	4	7	2	1	6	5	8	3	4	7	6	2	1	9	8	4	7	5	3	6
8	6	3	9	4	5	1	7	2	9	3	5	4	8	6	3	5	9	1	7	2
						4	2	1	8	5	9	7	6	3						
						8	3	9	1	6	7	5	2	4						
						6	5	7	3	4	2	1	9	8						
1	4	7	3	9	8	2	6	5	7	9	3	8	4	1	2	3	6	9	5	7
8	3	9	5	6	2	7	1	4	6	2	8	9	3	5	4	1	7	8	6	2
2	5	6	7	1	4	3	9	8	5	1	4	6	7	2	8	5	9	3	4	1
6	7	2	1	5	9	8	4	3				5	1	4	6	9	2	7	8	3
5	1	3	8	4	7	6	2	9				3	6	9	7	8	5	2	1	4
4	9	8	2	3	6	1	5	7				2	8	7	3	4	1	6	9	5
3	8	5	9	2	1	4	7	6				1	2	6	5	7	8	4	3	9
9	6	1	4	7	3	5	8	2				7	9	3	1	6	4	5	2	8
7	2	4	6	8	5	9	3	1				4	5	8	9	2	3	1	7	6

Multiplication by Four Digit Multiplier

Step 1 : The right most digit of multiplicand is multiplied by right most digit of multiplier.

Step 2 : For the right most pair of multiplicand and right most pair of multiplier apply inside-outside principle (plus carry).

Step 3 : For the right most triplet of multiplicand and right most triplet of multiplier apply inside-outside principle (plus carry).

Step 4 : For each successive quadruple of multiplicand and multiplier apply inside-outside principle (plus carry).

Step 5 : For the left most triplet of multiplicand and left most triplet of multiplier apply inside-outside principle (plus carry).

Step 6 : For the left most pair of multiplicand and left most pair of multiplier apply inside-outside principle (plus carry.)

Step 7 : For left most digit of multiplicand is multiplied by left most digit of multiplier (plus carry)

Example : Multiply 987021 by 6543.

$$987021 \times 6543$$

Step 1 : Multiply 1 by 3;

$1 \times 3 = 3$;

write 3

$$\begin{array}{r} 987021 \times 6543 \\ \hline 3 \end{array}$$

Step 2 : For pair (21) and (43) apply inside-outside principle

$[(1 \times 4) + (2 \times 3)] = 4 + 6 = 10$;

write 0, carry over 1

$$\begin{array}{r} 987021 \times 6543 \\ \hline 03 \end{array}$$

Step 3 : For triplet (021) and (543) apply inside-outside principle + (carry 1)

$[(1 \times 5) + (2 \times 4) + (0 \times 3)] + 1$

$= 5 + 8 + 0 + 1 = 14$;

write 4, carry over 1

$$\begin{array}{r} 987021 \times 6543 \\ \hline 403 \end{array}$$

Step 4 :

- For quadruple (7021) and multiplier (6543) apply inside-outside principle + (carry 1) ;
  $[(1 \times 6) + (2 \times 5) + (0 \times 4) + (7 \times 3)] + 1$
  $= 6 + 10 + 0 + 21 + 1 = 38$;
  write 8, carry over 3

$$\begin{array}{r} 987021 \times 6543 \\ \hline 8403 \end{array}$$

- For quadruple (8702) and multiplier (6543) apply inside-outside principle + (carry 3)
  $[(2 \times 6) + (0 \times 5) + (7 \times 4) + (8 \times 3)] + 3$
  $= 12 + 0 + 28 + 24 + 3 = 67$;
  write 7, carry over 6

$$\begin{array}{r} 987021 \times 6543 \\ \hline 78403 \end{array}$$

- For quadruple (9870) and multiplier (6543) apply inside-outside principle + (carry 6)
  $[(0 \times 6) + (7 \times 5) + (8 \times 4) + (9 \times 3)] + 6$
  $= 0 + 35 + 32 + 27 + 6 = 100$;
  write 0, carry over 10

$$\begin{array}{r} 987021 \times 6543 \\ \hline 078403 \end{array}$$

Step 5 : For triplet (987) and (654) apply inside-outside principle + (carry 10)

$[(7 \times 6) + (8 \times 5) + (9 \times 4)] + 10$

$= 42 + 40 + 36 + 10 = 128$;

write 8, carry over 12.

$$\begin{array}{r} 987021 \times 6543 \\ \hline 8078403 \end{array}$$

Step 6 : For pair (98) and (65) apply inside-outside principle + (carry 12)

$[(8 \times 6) + (9 \times 5)] \times 12$

$= 48 + 45 + 12 = 105$;

write 5, carry over 10.

$$\begin{array}{r} 987021 \times 6543 \\ \hline 58078403 \end{array}$$

Step 7 : Multiply 9 by 6 + (carry 10)

$9 \times 6 = 54 + 10 = 64$;

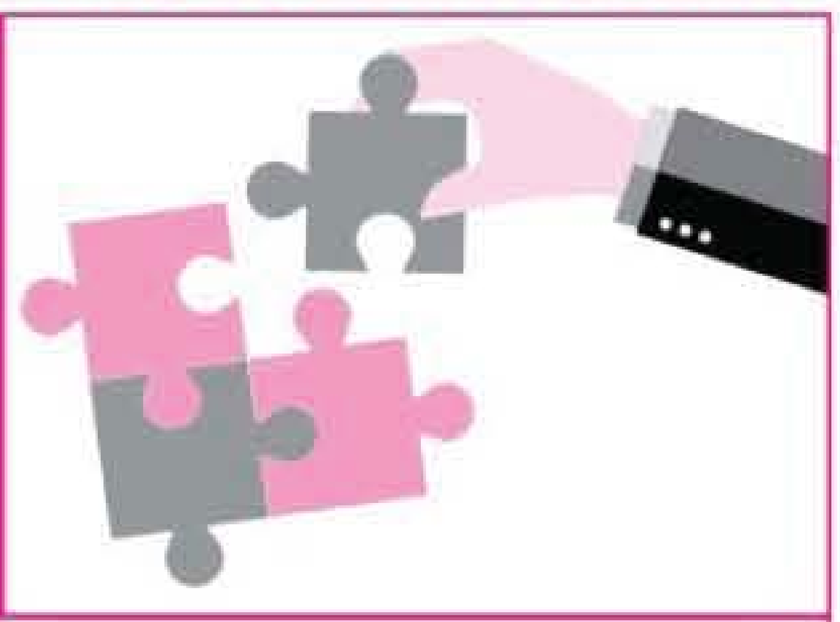
write 64

The answer is 6458078403

$$\begin{array}{r} 987021 \times 6543 \\ \hline 6458078403 \end{array}$$

◆◆

PUZZLE CORNER

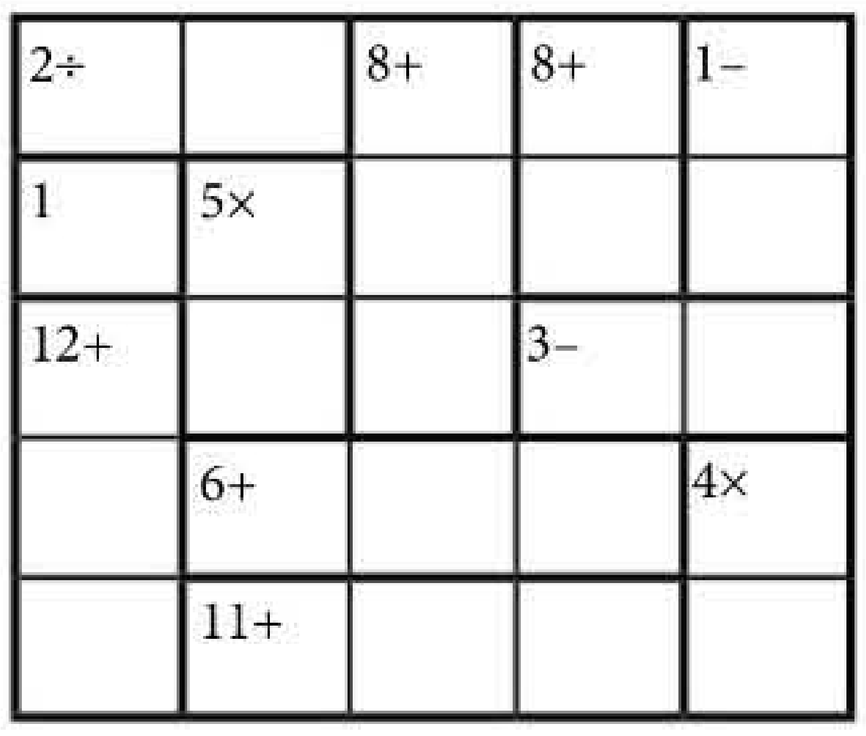


MATHDOKU

Introducing MATHDOKU, a mixture of ken-ken, sudoku and Mathematics.

In this puzzle 5×5 grid is given, your objective is to fill the digits 1-5 so that each appear exactly once in each row and each column.

Notice that most boxes are part of a cluster. In the upper-left corner of each multibox cluster is a value that is combined using a specified operation on its numbers. For example, if that value is 3 for a two-box cluster and operation is multiply, you know that only 1 and 3 can go in there. But it is your job to determine which number goes where! A few cluster may have just one box and that is the number that fills that box.



2÷		8+	8+	1−
1	5×			
12+			3−	
	6+			4×
	11+			

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

CBSE warm-up!

CLASS-XII

Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2019-20.

Series 4 Continuity & Differentiability, Application of Derivatives

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short-answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long-answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION - A

1. Let f be defined on $[-5, 5]$ as

$f(x)=\begin{cases} x, \text{if } x \text{ is rational} \\ -x, \text{ if } x \text{ is irrational}\end{cases}$, then $f(x)$ is

(a) continuous at every x except $x = 0$
(b) discontinuous at every x except $x = 0$
(c) continuous everywhere
(d) discontinuous everywhere

2. If $f(x) = x^3 - 6x^2 + 9x + 3$ be a decreasing function, then x lies in

(a) $(-\infty, -1) \cap (3, \infty)$ (b) $(1, 3)$
(c) $(3, \infty)$ (d) None of these

3. $\dfrac{d}{dx}\left[\tan^{-1}\left(\dfrac{a-x}{1+ax}\right)\right]$ is equal to

(a) $-\dfrac{1}{1+x^2}$ (b) $\dfrac{1}{1+a^2}-\dfrac{1}{1+x^2}$

(c) $\dfrac{1}{1+\left(\dfrac{a-x}{1+ax}\right)^2}$ (d) $\dfrac{-1}{\sqrt{1-\left(\dfrac{a-x}{1+ax}\right)^2}}$

4. The slope of the tangent to the curve $x = a\sin t$, $y=a\left\{\cos t+\log\left(\tan\dfrac{t}{2}\right)\right\}$ at the point 't' is

(a) $\tan t$ (b) $\cot t$
(c) $\tan(t/2)$ (d) None of these

5. If $y=e^{\frac{1}{2}\log(1+\tan^2 x)}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{1}{2}\sec^2 x$ (b) $\sec^2 x$

(c) $\sec x \tan x$ (d) $e^{\frac{1}{2}\log(1+\tan^2 x)}$

6. The function $f(x)=x+\dfrac{4}{x}$ has

(a) a local maxima at $x = 2$ and local minima at $x = -2$
(b) a local minima at $x = 2$ and local maxima at $x = -2$
(c) absolute maxima at $x = 2$ and absolute minima at $x = -2$
(d) absolute minima at $x = 2$ and absolute maxima at $x = -2$

7. Let $f(x)$ and $g(x)$ be differentiable functions for $0 \le x \le 1$, such that $f(0) = 2$, $g(0) = 0$, $f(1) = 6$. Let there exist a real number c in $(0, 1)$ such that $f'(c) = 2g'(c)$, then the value of $g(1)$ must be

(a) 1 (b) 2 (c) -2 (d) -1

8. The distance 's' metres covered by a body in t seconds, is given by $s = 3t^2 - 8t + 5$. The body will stop after

(a) 1 second (b) 3/4 second

(c) 4/3 seconds (d) 4 seconds

9. If $f(x)=\begin{cases}\frac{1}{2}(x-|x|), & \text{when } x \neq 0\\ 2, & \text{when } x=0\end{cases}$, then at $x=0$

(a) $\lim\limits_{x\to 0^-} f(x)=1$ (b) f is continuous

(c) f is discontinuous (d) None of these

10. Differentiate e^x w.r.t. $\sqrt{x}$.

(a) $-2e^x\sqrt{x}$ (b) $-e^x\sqrt{x}$

(c) $e^x\sqrt{x}$ (d) $2e^x\sqrt{x}$

11. Prove that the function e^{2x} is strictly increasing on R.

12. Find the derivative of $\sin(\sin x^2)$ at $x=\sqrt{\frac{\pi}{2}}$.

13. Discuss the continuity of the function $f(x) = \sin x - \cos x$.

14. Differentiate $\frac{8^x}{x^8}$ w.r.t. x.

15. The total revenue received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$.

16. Differentiate $\cos(\log(\sin x))$ w.r.t. x.

17. Show that the function $x^3 - 3x^2 + 6x + 3$ has neither a maximum nor a minimum.

18. Differentiate $e^{x^5}\cdot e^x$ w.r.t. x.

19. The radius of a circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?

20. Find the intervals in which the function $\sin 3x$, $0 \le x \le \frac{\pi}{2}$ is increasing.

SECTION - B

21. Find the value of the constant k, so that the function

$$f(x)=\begin{cases}\dfrac{1-\cos 2x}{2x^2}, & x\neq 0\\ k, & x=0\end{cases}$$ is continuous at $x = 0$.

22. Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$.

23. If $f(a) = 2$, $f'(a) = 1$, $g(a) = -1$, $g'(a) = 2$, then find the value of $\lim\limits_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$.

OR

If $x^y = e^{x-y}$, then prove that $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$.

24. Is the function $f(x)=\begin{cases}\dfrac{1-x^n}{1-x}, & x\neq 1\\ n-1, & x=1\end{cases}$, $n\in N$ is continuous at $x = 1$?

25. Find the equation of the tangent to the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at the point $x = 1$.

OR

Determine the intervals in which the function $f(x)=x^4-\frac{4x^3}{3}$ is strictly increasing or strictly decreasing.

26. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\ldots \text{ to } \infty}}}$, then prove that $\frac{dy}{dx}=\frac{\cos x}{2y-1}$.

SECTION - C

27. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

28. Find the difference between the greatest and least values of the function $f(x) = \sin 2x - x$ on $[-\pi/2, \pi/2]$.

OR

Find the equation of the tangent to the curve $y=\frac{x-7}{(x-2)(x-3)}$ at the point, where it cuts the x-axis.

29. If $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$, differentiate $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ with respect to $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

OR

If $y^2 = P(x)$, a polynomial of degree $n > 3$, then show that $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)=P(x)P'''(x)$.

30. It is given that for the function f given by $f(x) = x^3 + bx^2 + ax$, $x \in [1, 3]$ rolle's theorem holds with $c=2+\frac{1}{\sqrt{3}}$. Find the values of a and b.

31. Water is dripping out from a conical funnel of semi-vertical angle $\pi/4$ at the uniform rate of 2 cm^2/sec in 4 cm, find the rate of decrease of the slant height of water.

32. If $y=(\sin x-\cos x)^{\sin x-\cos x}, \frac{\pi}{4}<x<\frac{3\pi}{4}$, then find $\frac{dy}{dx}$.

SECTION - D

33. If the function f defined by

$$f(x)=\begin{cases} x+a\sqrt{2}\sin x, 0\le x<\dfrac{\pi}{4} \\ 2x\cot x+b, \dfrac{\pi}{4}\le x\le\dfrac{\pi}{2} \\ a\cos 2x-b\sin x, \dfrac{\pi}{2}<x\le\pi \end{cases}$$

is continuous in $[0, \pi]$, then find the values of a and b.

34. A jet of an enemy is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). What is the shortest distance between the soldier and the jet?

OR

The sum of the perimeters of a circle and a square is k, where k is some constant. Prove that the sum of their areas is least when the side of the square is equal to the diameter of the circle.

35. If $y^2 = a^2\cos^2 x + b^2\sin^2 x$, then prove that $\frac{d^2y}{dx^2}+y=\frac{a^2b^2}{y^3}$.

36. Find the intervals in which the function f given by $f(x) = \sin x - \cos x, 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

OR

If $y=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$, then prove that $\frac{dy}{dx}=\frac{1}{a+b\cos x}$, $a > b > 0$.

SOLUTIONS

1. (b) : As $x \to 0$ both x and $-x$ tend to zero, $f(0) = 0$

$\therefore$ $f(x)$ is continuous at $x = 0$.

For $x \ne 0$, $x \ne -x$, so $f(x)$ is discontinuous at every x except $x = 0$.

2. (b) : Given, $f(x) = x^3 - 6x^2 + 9x + 3$

$\therefore\ f'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

For decreasing, $f'(x) < 0$

$\Rightarrow (x - 3)(x - 1) < 0 \ \therefore\ x \in (1, 3)$

3. (a) : We have, $\frac{d}{dx}\left[\tan^{-1}\left(\frac{a-x}{1+ax}\right)\right]$

$$=\frac{d}{dx}[\tan^{-1}a-\tan^{-1}x]=0-\frac{1}{1+x^2}=-\frac{1}{1+x^2}$$

4. (b) : Slope of the tangent $=\frac{dy}{dx}=\left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right)$

$$=\frac{a(-\sin t+\operatorname{cosec} t)}{a\cos t}=\cot t.$$

5. (c) : Given, $y=e^{\frac{1}{2}\log(1+\tan^2 x)}=(\sec^2 x)^{1/2}=\sec x$

$\therefore\quad \frac{dy}{dx}=\sec x\tan x$

6. (b) : Given, $f(x) = x+\frac{4}{x}$ $\quad\therefore\quad f'(x)=1-\frac{4}{x^2}$

Now, $f'(x) = 0 \Rightarrow x = \pm 2$

Now, $f''(x)=\frac{8}{x^3}>0$ for $x = 2$ and < 0 for $x = -2$

$\therefore$ $f(x)$ has local minima at $x = 2$ and local maxima at $x = -2$.

7. (b) : $\frac{f'(c)}{g'(c)}=\frac{f(1)-f(0)}{g(1)-g(0)}\Rightarrow 2=\frac{6-2}{g(1)-0}\Rightarrow g(1)=2$

8. (c) : Given, distance $(s) = 3t^2 - 8t + 5$ and

Velocity (v) = $\frac{ds}{dt}=6t-8$

The body will be stopped when velocity is zero.

$\Rightarrow\ 6t-8=0\ \Rightarrow\ t=\frac{4}{3}$ seconds

9. (c) : Here, $f(0) = 2$

Now, $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\frac{x-|x|}{2}=\lim\limits_{x\to 0^-}\frac{x+x}{2}=0$

Since $\lim\limits_{x\to 0^-} f(x)\ne f(0)$

$\therefore$ f is discontinuous at $x = 0$.

10. (d) : Let $u = e^x$ and $v=\sqrt{x}$.

Then, $\frac{du}{dx}=e^x$ and $\frac{dv}{dx}=\frac{1}{2}x^{-1/2}=\frac{1}{2\sqrt{x}}$

$\therefore\quad \frac{du}{dv}=\frac{(du/dx)}{(dv/dx)}=\frac{e^x}{(1/2\sqrt{x})}=2e^x\sqrt{x}$

11. Let $f(x) = e^{2x}$ and domain of $f = R$.

Differentiating w.r.t. x, we get

$f'(x) = e^{2x}\cdot 2 > 0$ for all $x \in R$

$\Rightarrow$ $f(x)$ is strictly increasing on R.

12. Let $f(x) = \sin(\sin x^2)$ ...(i)

Differentiating (i) w.r.t. x, we get

$f'(x) = \cos(\sin x^2)\cdot\cos x^2\cdot 2x$

$\therefore\quad f'\left(\sqrt{\frac{\pi}{2}}\right)=\cos\left(\sin\frac{\pi}{2}\right)\cdot\cos\frac{\pi}{2}\cdot\cos\frac{\pi}{2}\cdot 2\cdot\sqrt{\frac{\pi}{2}}=0$

13. Since, $\sin x$ and $\cos x$ are continuous functions and difference of two continuous functions is also a continuous function.

$\therefore$ $f(x) = \sin x - \cos x$ is a continuous function.

14. Let $y = \dfrac{8^x}{x^8}$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{x^8 \cdot 8^x \cdot \log 8 - 8^x \cdot 8x^7}{(x^8)^2} = \frac{x^7 8^x (x \log 8 - 8)}{x^{16}}$$

$$= \frac{8^x (x \log 8 - 8)}{x^9} = \frac{8^x}{x^8}\left(\log 8 - \frac{8}{x}\right).$$

15. Given, $R(x) = 13x^2 + 26x + 15$

$\therefore$ Marginal Revenue is given by $MR(x) = \dfrac{dR}{dx}$

$= 26x + 26$

At $x = 7$, $MR(7) = 26 \times 7 + 26 = 26(7 + 1) = 26 \times 8 = 208$

16. Let $y = \cos(\log(\sin x))$

Then, $\dfrac{dy}{dx} = \dfrac{d}{dx}\cos(\log(\sin x))$

$= -\sin(\log(\sin x)) \cdot \dfrac{1}{\sin x} \cdot \cos x = -\cot x \sin(\log(\sin x))$

17. Let $y = x^3 - 3x^2 + 6x + 3$

$\therefore$ $\dfrac{dy}{dx} = 3(x^2 - 2x + 2) = 3\{(x - 1)^2 + 1\} > 0$

Hence, y is an increasing function in its domain $(-\infty, \infty)$

Thus, y has neither a maximum nor a minimum value.

18. Let $y = e^{x^5} \cdot e^x = e^{x^5 + x}$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(e^{x^5+x}) = e^{x^5+x} \cdot \frac{d}{dx}(x^5 + x)$$

$$= e^{x^5+x} \cdot (5x^4 + 1)$$

19. Let C and r be the circumference and radius of the circle respectively at any time t.

Then, $C = 2\pi r \Rightarrow \dfrac{dC}{dt} = 2\pi \dfrac{dr}{dt}$

$$\therefore \quad \frac{dC}{dt} = 2\pi \times 0.7 = 2 \times \frac{22}{7} \times \frac{7}{10} = 4.4 \text{ cm/sec}$$

$$\left[\because \frac{dr}{dt} = 0.7 \text{ cm/sec (Given)}\right]$$

20. Let $f(x) = \sin 3x \Rightarrow f'(x) = 3\cos 3x$

Now, $3\cos 3x > 0 \Rightarrow \cos 3x > 0 \Rightarrow 0 \le 3x \le \dfrac{\pi}{2}$

$$\Rightarrow \quad 0 \le x \le \frac{\pi}{6} \quad \Rightarrow x \in \left[0, \frac{\pi}{6}\right].$$

21. It is given that the function $f(x)$ is continuous at $x = 0$.

$$\therefore \quad \lim_{x\to 0} f(x) = f(0) \quad \Rightarrow \quad \lim_{x\to 0} f(x) = k$$

$$\Rightarrow \quad \lim_{x\to 0} \frac{1 - \cos 2x}{2x^2} = k \quad \Rightarrow \quad \lim_{x\to 0} \frac{2\sin^2 x}{2x^2} = k$$

$$\Rightarrow \quad \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 = k \quad \Rightarrow \quad k = 1$$

Thus, $f(x)$ is continuous at $x = 0$, if $k = 1$.

22. Given, $f(x) = 4x^2 + 5x + 2$

Differentiating w.r.t x, we get $f'(x) = 8x + 5$

We know that $f(x + \delta x) \approx f(x) + f'(x)\delta x$ ...(i)

Now, $x = 2$ and $x + \delta x = 2.01$, then $\delta x = 0.01$

$\therefore$ From (i), we have

$f(2.01) \approx f(2) + f'(2) \times 0.01$

$= 4 \times 2^2 + 5 \times 2 + 2 + (8 \times 2 + 5) \times 0.01$

$= 28 + 21 \times 0.01 = 28 + 0.21 = 28.21$

Hence, the approximate value of $f(2.01)$ is 28.21.

23. We have, $\displaystyle\lim_{x\to a} \frac{g(x)f(a) - g(a)f(x)}{x - a}$

$$= \lim_{x\to a} \frac{[g(x)f(a) - g(a)f(a)] - [g(a)f(x) - g(a)f(a)]}{x - a}$$

$$= f(a)\lim_{x\to a}\left(\frac{g(x) - g(a)}{x - a}\right) - g(a)\lim_{x\to a}\left(\frac{f(x) - f(a)}{x - a}\right)$$

$$= f(a)\lim_{h\to 0}\frac{g(a+h) - g(a)}{h} - g(a)\lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

[Putting $x = a + h$]

$= f(a)\, g'(a) - g(a)\, f'(a) = 2\cdot 2 - (-1)\cdot 1 = 5$

OR

Given, $x^y = e^{x-y}$

$\Rightarrow$ $e^{y\log x} = e^{x-y}$ $\quad [\because x^y = e^{\log x^y} = e^{y\log x}]$

$\Rightarrow$ $y\log x = x - y \Rightarrow y\log x + y = x$

$\Rightarrow$ $y(1 + \log x) = x \Rightarrow y = \dfrac{x}{1 + \log x}$ ...(i)

Differentiating both sides of (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1 + \log x) \times 1 - x\left(0 + \frac{1}{x}\right)}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$

24. Given, $f(x) = n - 1$ when $x = 1 \Rightarrow f(1) = n - 1$

Also, $\displaystyle\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{1 - x^n}{1 - x} = \lim_{x\to 1}\frac{x^n - 1^n}{x - 1} = n \cdot 1^{n-1} = n$

For continuity at $x = 1$, $\displaystyle\lim_{x\to 1} f(x) = f(1)$

$\Rightarrow$ $n = n - 1 \Rightarrow -1 = 0$, which is not possible.

$\therefore$ $\displaystyle\lim_{x\to 1} f(x) \ne f(1) \Rightarrow f$ is discontinuous.

25. Here, $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ ...(i)

When $x = 1$, $y = 1 - 6 + 13 - 10 + 5 = 3$.

We want to find tangent to (i) at $P(1, 3)$.

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_P = 4 - 18 + 26 - 10 = 2$$

Hence the equation of the tangent to (i) at $P(1, 3)$ is
$y - 3 = 2(x - 1) \Rightarrow y = 2x + 1$

OR

Given, $f(x) = x^4 - \frac{4x^3}{3}, D_f = R.$

Differentiating w.r.t. x, we get

$$f'(x) = 4x^3 - \frac{4}{3} \cdot 3x^2 = 4x^2(x-1)$$

For $f(x)$ to be increasing, we must have
$f'(x) > 0 \Rightarrow 4x^2(x - 1) > 0$
$\Rightarrow x - 1 > 0$ $\quad (\because x^2 > 0, x \neq 0)$
$\Rightarrow x > 1 \Rightarrow x \in (1, \infty)$
So, $f(x)$ is strictly increasing in $(1, \infty)$.
Similarly, for $f(x)$ to be decreasing, we must have
$f'(x) < 0 \Rightarrow 4x^2(x - 1) < 0 \Rightarrow x - 1 < 0$
$\Rightarrow x < 1 \Rightarrow x \in (-\infty, 1)$.
So, $f(x)$ is strictly decreasing in $(-\infty, 1)$.

26. Let $y = \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}$ $\therefore y = \sqrt{\sin x + y}$
$\Rightarrow y^2 = \sin x + y$ $\quad$ [Squaring both sides]
Differentiating w.r.t. x on both sides, we get

$$2y\frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}(2y - 1) = \cos x \Rightarrow \frac{dy}{dx} = \frac{\cos x}{2y - 1}.$$

27. We have, $f(x) = |x|^3 = \begin{cases} x^3, & \text{if } x \geq 0 \\ (-x)^3 = -x^3, & \text{if } x < 0 \end{cases}$

Now, $\lim\limits_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^-} \frac{-x^3 - 0}{x} = 0$

Also, $\lim\limits_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^+} \frac{x^3 - 0}{x} = 0$

Since $f'(0^-) = f'(0^+)$

So, $f(x)$ is differentiable at $x = 0$ and the derivative of $f(x)$ is given by $f'(x) = \begin{cases} 3x^2, & \text{if } x \geq 0 \\ -3x^2, & \text{if } x < 0 \end{cases}$

Now, $\lim\limits_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim\limits_{x \to 0^-} \frac{-3x^2 - 0}{x} = 0$

Also, $\lim\limits_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim\limits_{x \to 0^+} \frac{3x^2 - 0}{x - 0} = 0$

Since $f''(0^-) = f''(0^+)$, therefore $f'(x)$ is differentiable at $x = 0$.

Hence, $f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$.

28. We have, $f(x) = \sin 2x - x$
$\therefore f'(x) = 2\cos 2x - 1$
At stationary points, put $f'(x) = 0 \Rightarrow 2\cos 2x - 1 = 0$

$$\Rightarrow \cos 2x = \frac{1}{2} \Rightarrow 2x = -\frac{\pi}{3}, \frac{\pi}{3} \Rightarrow x = -\frac{\pi}{6}, \frac{\pi}{6}.$$

Now find the values of $f(x)$ at these stationary points and at $-\pi/2$ and $\pi/2$.
Since, $f(x) = \sin 2x - x$

$$\therefore \quad f\left(-\frac{\pi}{2}\right) = \sin(-\pi) + \frac{\pi}{2} = \frac{\pi}{2},$$

$$f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{3}\right) + \frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6},$$

$$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{3} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6},$$

$$f\left(\frac{\pi}{2}\right) = \sin\pi - \frac{\pi}{2} = -\frac{\pi}{2}$$

So, the greatest and the least values of $f(x)$ on $[-\pi/2, \pi/2]$ are $\pi/2$ and $-\pi/2$ respectively.

Hence, required difference $= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi.$

OR

Refer to answer 50, Page no. 135 of MTG CBSE Champion Mathematics, Class-12

29. Let $u = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$ and $v = \tan^{-1}\left(\frac{2x}{1 - x^2}\right)$
Putting $x = \tan\theta$, we have
$u = \tan^{-1}(\tan 3\theta)$ and $v = \tan^{-1}(\tan 2\theta)$.
$\Rightarrow u = 3\theta$ and $v = 2\theta$

$$\left[\begin{array}{l} \because -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < \tan\theta < \frac{1}{\sqrt{3}} \Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6} \\ \Rightarrow -\frac{\pi}{2} < 3\theta < \frac{\pi}{2} \text{ and } -\frac{\pi}{3} < 2\theta < \frac{\pi}{3} \end{array}\right]$$

$\Rightarrow u = 3\tan^{-1}x$ and $v = 2\tan^{-1}x$

$$\Rightarrow \frac{du}{dx} = \frac{3}{1 + x^2} \text{ and } \frac{dv}{dx} = \frac{2}{1 + x^2} \Rightarrow \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{3}{1+x^2}}{\frac{2}{1+x^2}} = \frac{3}{2}.$$

Monthly Test Drive-3 CLASS XII ANSWER KEY

1.	(d)	**2.**	(b)	**3.**	(d)	**4.**	(c)	**5.**	(d)
6.	(d)	**7.**	(a,c)	**8.**	(a,b,c)	**9.**	(b)	**10.**	(a,b)
11.	(a,b,c)	**12.**	(b,c,d)	**13.**	(a,b,c,d)	**14.**	(a)	**15.**	(b)
16.	(b)	**17.**	(10)	**18.**	(144)	**19.**	(0)	**20.**	(896)

OR

Given, $y^2 = P(x)$ ...(i)

Differentiating (i) w.r.t. x, we get

$$2y\frac{dy}{dx} = P'(x) \qquad \text{...(ii)}$$

Differentiating again w.r.t. x, we get

$$2y\cdot\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\cdot\frac{dy}{dx} = P''(x) \Rightarrow 2y\frac{d^2y}{dx^2} = P''(x) - 2\left(\frac{dy}{dx}\right)^2$$

Multiplying both sides by y^2, we have

$$2y^3\frac{d^2y}{dx^2} = y^2P''(x) - 2y^2\left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad 2y^3\frac{d^2y}{dx^2} = y^2P''(x) - \frac{1}{2}[P'(x)]^2 \qquad \text{[Using (ii)]}$$

Differentiating again w.r.t. x, we get

$$2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = 2y\frac{dy}{dx}P''(x) + y^2P'''(x) - \frac{1}{2}2P'(x)\cdot P''(x)$$

$= P'(x)\,P''(x) + y^2\,P'''(x) - P'(x)\,P''(x)$ [Using (ii)]

$= y^2\,P'''(x) = P(x)\,P'''(x)$ [Using (i)]

30. Since, it is given that the Rolle's theorem holds for $f(x) = x^3 + bx^2 + ax$ defined on $[1, 3]$ with $c = 2 + \frac{1}{\sqrt{3}}$.

$\therefore \quad f(1) = f(3)$ and $f'(c) = 0$

$\Rightarrow \quad 1 + b + a = 27 + 9b + 3a$ and $3c^2 + 2bc + a = 0$

$$\Rightarrow \quad 2a + 8b + 26 = 0 \text{ and } 3\left(2+\frac{1}{\sqrt{3}}\right)^2 + 2b\left(2+\frac{1}{\sqrt{3}}\right) + a = 0$$

$$\Rightarrow \quad a + 4b + 13 = 0 \text{ and } a + 4b + 13 + \frac{2}{\sqrt{3}}(b+6) = 0$$

$$\Rightarrow \quad a + 4b + 13 = 0 \text{ and } 0 + \frac{2}{\sqrt{3}}(b+6) = 0$$

$\Rightarrow \quad a + 4b + 13 = 0$ and $b = -6 \quad \Rightarrow \quad a = 11$ and $b = -6$

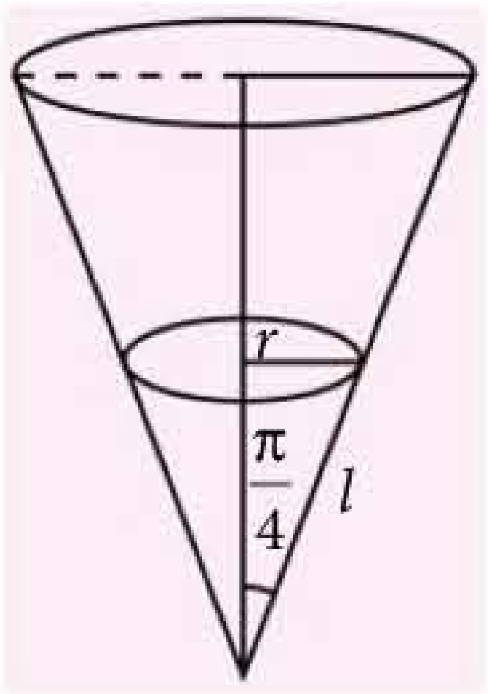
31. At any time t, water forms a right circular cone of semi-vertical angle $\pi/4$.

Let l cm be the slant height and r cm be the radius of the water cone at that instant, then

$$r = l\sin\frac{\pi}{4} = \frac{l}{\sqrt{2}} \qquad \text{...(i)}$$

Let S be the (curved) surface area of the water cone, then

$$S = \pi rl = \frac{\pi}{\sqrt{2}}l^2 \qquad \text{... (ii) [Using(i)]}$$



Differentiating (ii) w.r.t. t, we get

$$\frac{dS}{dt} = \frac{\pi}{\sqrt{2}}.2l\frac{dl}{dt}$$

It is given that water is leaking from the funnel at the rate of 2 cm^2/sec.

$$\therefore \ \frac{dS}{dt} = -\,2\text{cm}^2/\text{sec}\left(\frac{dS}{dt} \text{ is } -\text{ve, since } S \text{ is decreasing}\right)$$

$$\Rightarrow \quad -2 = \frac{\pi}{\sqrt{2}}.2l\frac{dl}{dt} \Rightarrow \frac{dl}{dt} = -\frac{\sqrt{2}}{\pi l}$$

$$\text{At } l = 4 \text{ cm}, \frac{dl}{dt} = -\frac{\sqrt{2}}{\pi\times 4}\text{cm/sec}$$

Hence, the slant height of the cone is decreasing at the rate of $\frac{\sqrt{2}}{4\pi}$ cm/sec.

32. *Refer to answer 64, Page no. 110 of MTG CBSE Champion Mathematics, Class-12*

33. Since, the given function $f(x)$ is continuous in $[0, \pi]$, so it must be continuous at $x = \pi/4$ and$x = \pi/2$

$$\text{Now,} \quad \lim_{x\to\frac{\pi}{4}^-} f(x) = f\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}^+} f(x)$$

$$\Rightarrow \lim_{x\to\frac{\pi}{4}^-}(x + a\sqrt{2}\sin x) = 2.\frac{\pi}{4}\cot\frac{\pi}{4} + b = \lim_{x\to\frac{\pi}{4}^+}(2x\cot x + b)$$

$$\Rightarrow \quad \frac{\pi}{4} + a\sqrt{2}.\sin\frac{\pi}{4} = \frac{\pi}{2} + b = 2.\frac{\pi}{4}\cot\frac{\pi}{4} + b$$

$$\Rightarrow \quad \frac{\pi}{4} + a\sqrt{2}.\frac{1}{\sqrt{2}} = \frac{\pi}{2} + b = \frac{\pi}{2}.1 + b$$

$$\Rightarrow \quad \frac{\pi}{4} + a = \frac{\pi}{2} + b \Rightarrow a - b = \frac{\pi}{4} \qquad \text{...(i)}$$

$$\text{Also,} \ \lim_{x\to\frac{\pi}{2}^-} f(x) = f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}^+} f(x)$$

$$\Rightarrow \quad \lim_{x\to\frac{\pi}{2}^-}(2x\cot x + b) = 2.\frac{\pi}{2}\cot\frac{\pi}{2} + b$$

$$= \lim_{x\to\frac{\pi}{2}^+}(a\cos 2x - b\sin x)$$

$$\Rightarrow \quad 2.\frac{\pi}{2}\cot\frac{\pi}{2} + b = \pi\cdot 0 + b = a\cos\pi - b\sin\frac{\pi}{2}$$

$\Rightarrow \ \pi\cdot 0 + b = b = a\cdot(-1) - b\cdot 1 \quad \Rightarrow \quad a = -2b$ ...(ii)

From (i) and (ii), we have

$$-2b - b = \frac{\pi}{4} \Rightarrow b = -\frac{\pi}{12} \text{ and } a = \frac{\pi}{6}$$

Hence, the given function f is continuous in $[0, \pi]$, if $a = \frac{\pi}{6}$ and $b = -\frac{\pi}{12}$.

34. Let $P(x, y)$ be the position of jet and the soldier is placed at $A(3, 2)$. Then, the distance between the soldier and jet is given by

$AP=\sqrt{(x-3)^2+(y-2)^2}=\sqrt{(x-3)^2+x^4}$ $[\because y=x^2+2]$

Let $Z=(AP)^2$. Then, $Z=(x-3)^2+x^4$

Clearly, AP is maximum or minimum according as Z is maximum or minimum.

Now, $\frac{dZ}{dx}=2(x-3)+4x^3$ and $\frac{d^2Z}{dx^2}=12x^2+2$

For critical points put $\frac{dZ}{dx}=0$.

$\Rightarrow\ 2(x-3)+4x^3=0\ \Rightarrow\ 2x^3+x-3=0$

$\Rightarrow\ (x-1)(2x^2+2x+3)=0\ \Rightarrow\ x=1$

$[\because\ 2x^2+2x+3=0$ gives imaginary values of $x]$

Clearly, $\left(\frac{d^2Z}{dx^2}\right)_{x=1}=12+2=14>0$

Thus, Z is minimum when $x=1$. Putting $x=1$ in $y=x^2+2$, we get $y=3$. So, the coordinate of P are $(1, 3)$. Hence, AP is minimum when jet is at the point $(1, 3)$ on the curve.

$\therefore\ AP=\sqrt{(1-3)^2+1^2}=\sqrt{5}$

Hence, the shortest distance $=\sqrt{5}$.

OR

Refer to answer 86, Page no. 146 of MTG CBSE Champion Mathematics, Class-12

35. We have, $y^2=a^2\cos^2 x+b^2\sin^2 x$

$\Rightarrow\ 2y^2=a^2(2\cos^2 x)+b^2(2\sin^2 x)$ [Multiply both sides by 2]

$\Rightarrow\ 2y^2=a^2(1+\cos 2x)+b^2(1-\cos 2x)$

$\Rightarrow\ 2y^2=(a^2+b^2)+(a^2-b^2)\cos 2x$ ...(i)

Differentiating (i) w.r.t. x, we get

$$4y\frac{dy}{dx}=-2(a^2-b^2)\sin 2x$$

$$\Rightarrow\ 2y\frac{dy}{dx}=-(a^2-b^2)\sin 2x \qquad \text{...(ii)}$$

Also, from (i), we have

$2y^2-(a^2+b^2)=(a^2-b^2)\cos 2x$ ...(iii)

Squaring (ii) and (iii) and adding, we get

$$4y^2\left(\frac{dy}{dx}\right)^2+\{2y^2-(a^2+b^2)\}^2=(a^2-b^2)^2\{\sin^2 2x+\cos^2 2x\}$$

$$\Rightarrow\ 4y^2\left(\frac{dy}{dx}\right)^2+4y^4-4y^2(a^2+b^2)+(a^2+b^2)^2=(a^2-b^2)^2$$

$$\Rightarrow\ 4y^2\left\{\left(\frac{dy}{dx}\right)^2+y^2-(a^2+b^2)\right\}=(a^2-b^2)^2-(a^2+b^2)^2$$

$$\Rightarrow\ 4y^2\left\{\left(\frac{dy}{dx}\right)^2+y^2-(a^2+b^2)\right\}=-4a^2b^2$$

$$\Rightarrow\ \left(\frac{dy}{dx}\right)^2+y^2-(a^2+b^2)=-\frac{a^2b^2}{y^2}$$

Differentiating both sides again w.r.t. x, we get

$$2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+2y\frac{dy}{dx}=\frac{2a^2b^2}{y^3}\frac{dy}{dx}\ \Rightarrow\ \frac{d^2y}{dx^2}+y=\frac{a^2b^2}{y^3}$$

36. *Refer to answer 31, Page no. 131 of MTG CBSE Champion Mathematics, Class-12*

OR

Given, $y=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=\frac{2}{\sqrt{a^2-b^2}}\cdot\frac{1}{1+\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)^2}\cdot\sqrt{\frac{a-b}{a+b}}\cdot\sec^2\frac{x}{2}\cdot\frac{1}{2}$$

$$=\frac{\sqrt{a-b}}{\sqrt{a+b}\sqrt{a^2-b^2}}\cdot\frac{(a+b)\sec^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

$$=\frac{\sec^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}}=\frac{1}{(a+b)\cos^2\frac{x}{2}+(a-b)\sin^2\frac{x}{2}}$$

$\left[\text{Dividing numerator and denominator by }\sec^2\frac{x}{2}\right]$

$$=\frac{1}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}=\frac{1}{a+b\cos x}.$$

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MONTHLY TEST DRIVE

XII

This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 3: Matrices and Determinants

Time Taken : 60 Min.

Only One Option Correct Type

1. If $A=\begin{bmatrix}\dfrac{-1+i\sqrt{3}}{2i} & \dfrac{-1-i\sqrt{3}}{2i}\\ \dfrac{1+i\sqrt{3}}{2i} & \dfrac{1-i\sqrt{3}}{2i}\end{bmatrix}$, $i=\sqrt{-1}$ and $f(x) = x^2 + 2$, then $f(A)$ equals

(a) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (b) $\left(\dfrac{3-i\sqrt{3}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

(c) $\left(\dfrac{5-i\sqrt{3}}{2}\right)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ (d) $(2+i\sqrt{3})\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

2. Let $\Delta(x)=\begin{vmatrix}(x-2) & (x-1)^2 & x^3\\ (x-1) & x^2 & (x+1)^3\\ x & (x+1)^2 & (x+2)^3\end{vmatrix}$, then the coefficient of x in $\Delta(x)$ is

(a) -3 (b) -2 (c) -1 (d) 0

3. If $y = \sin mx$, then the value of the determinant $\begin{vmatrix}y & y_1 & y_2\\ y_3 & y_4 & y_5\\ y_6 & y_7 & y_8\end{vmatrix}$, where $y_n=\dfrac{d^n y}{dx^n}$, is

(a) m^9 (b) m^2 (c) m^3 (d) 0

4. Let $A+2B=\begin{bmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{bmatrix}$ and $2A-B=\begin{bmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{bmatrix}$.

Then $\text{tr}(A) - \text{tr}(B)$ has the value equal to

(a) 0 (b) 1 (c) 2 (d) 3

5. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is

(a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13}

6. Let $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ and $B=\begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix}$. If $10A^{10}$ + $\text{adj}(A^{10}) = B$, then $b_1 + b_2 + b_3 + b_4$ is equal to

(a) 91 (b) 92 (c) 111 (d) 112

One or More Than One Option(s) Correct Type

7. If $\begin{pmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{pmatrix}\begin{pmatrix}1 & \tan\theta\\ -\tan\theta & 1\end{pmatrix}^{-1}=\begin{pmatrix}a & -b\\ b & a\end{pmatrix}$, then

(a) $a = \cos 2\theta$ (b) $a = 1$

(c) $b = \sin 2\theta$ (d) $b = -1$

8. If $f(x)=\begin{vmatrix}a & -1 & 0\\ ax & a & -1\\ ax^2 & ax & a\end{vmatrix}$, then $f(2x) - f(x)$ is divisible by

(a) x (b) a

(c) $2a + 3x$ (d) x^2

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9. If P is an orthogonal matrix and $Q = PAP^T$ and $x = P^T Q^{1000} P$, then x^{-1} is (where A is an involuntary matrix)

(a) A (b) I
(c) A^{1000} (d) None of these

10. If $A + B + C = \pi$, $e^{i\theta} = \cos\theta + i\sin\theta$ and

$$z = \begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix}, \text{ then}$$

(a) Re $(z) = 4$ (b) Im $(z) = 0$
(c) Re $(z) = -4$ (d) Im $(z) = -1$

11. The positive integral solutions of the equation

$$\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 30 \text{ are}$$

(a) (3, 1, 1) (b) (1, 3, 1)
(c) (1, 1, 3) (d) (–1, 1, 3)

12. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when $n =$

(a) 57 (b) 55 (c) 58 (d) 56

13. The system of equations $x - y\cos\theta + z\cos 2\theta = 0$, $-x\cos\theta + y - z\cos\theta = 0$, $x\cos 2\theta - y\cos\theta + z = 0$ has non trivial solution for θ equals

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{6}$ (c) $\dfrac{2\pi}{3}$ (d) $\dfrac{\pi}{12}$

Comprehension Type

If $x > m, y > n, z > r$ $(x, y, z > 0)$ such that $\begin{vmatrix} x & n & r \\ m & y & r \\ m & n & z \end{vmatrix} = 0.$

14. The value of $\dfrac{x}{x-m} + \dfrac{y}{y-n} + \dfrac{z}{z-r}$ is

(a) 2 (b) –4 (c) 0 (d) –1

15. The greatest value of $\dfrac{xyz}{(x-m)(y-n)(z-r)}$ is

(a) 27 (b) 8/27
(c) 64/27 (d) None of these

Matrix Match Type

16. Given, $\begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix}$

$= px^5 + qx^4 + rx^3 + sx^2 + tx + w.$

Match the following columns.

	Column-I		Column-II
P.	w is equal to	1.	3
Q.	t is equal to	2.	1
R.	$(p + r)$ is equal to	3.	–1
S.	$(q + s)$ is equal to	4.	0

	P	Q	R	S
(a)	1	2	3	4
(b)	2	1	3	2
(c)	3	1	4	2
(d)	4	2	3	4

Numerical Answer Type

17. The value of $\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$ is

18. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then the value of $64(\sin^2 A + \sin^2 B + \sin^2 C)$ must be

19. Let S be the set which contains all possible values of l, m, n, p, q, r for which

$$A = \begin{bmatrix} l^2-3 & p & 0 \\ 0 & m^2-8 & q \\ r & 0 & n^2-15 \end{bmatrix}$$ be a non singular idempotent matrix. Then the sum of all the elements of the set S is

20. Let $a_r = r({}^7C_r)$, $b_r = (7-r)({}^7C_r)$ and $A_r = \begin{pmatrix} a_r & 0 \\ 0 & b_r \end{pmatrix}$.

$A = \sum_{r=0}^{7} A_r = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, then the value of $a + b$ must be

◆

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SELF CHECK

No. of questions attempted ……
No. of questions correct ……
Marks scored in percentage ……

Check your score! If your score is

> 90%	**EXCELLENT WORK !**	You are well prepared to take the challenge of final exam.
90-75%	**GOOD WORK !**	You can score good in the final exam.
74-60%	**SATISFACTORY !**	You need to score more next time.
< 60%	**NOT SATISFACTORY!**	Revise thoroughly and strengthen your concepts.

CBSE warm-up!

CLASS-XI

Chapterwise Practice questions for CBSE Exams as per the latest pattern and marking scheme issued by CBSE for the academic session 2019-20.

Series 4 Permutations and Combinations, Binomial Theorem

Time Allowed : 3 hours
Maximum Marks : 80

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) This question paper contains **36** questions.
(iii) Question **1-20** in Section-A are very short-answer-objective type questions carrying **1** mark each.
(iv) Question **21-26** in Section-B are short-answer type questions carrying **2** marks each.
(v) Question **27-32** in Section-C are long-answer-**I** type questions carrying **4** marks each.
(vi) Question **33-36** in Section-D are long-answer-**II** type questions carrying **6** marks each.

SECTION - A

1. If $\frac{1}{8!}+\frac{1}{9!}=\frac{x}{10!}$, then find the value of x.
(a) 90 (b) 100 (c) 80 (d) 95

2. If $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$, then
(a) $\text{Re}(z)=0$ (b) $\text{Im}(z)=0$
(c) $\text{Re}(z)>0, \text{Im}(z)>0$ (d) $\text{Re}(z)>0, \text{Im}(z)<0$

3. Find the value of n such that $\frac{{}^nP_4}{{}^{n-1}P_4}=\frac{5}{3}$, $n>4$.
(a) 11 (b) 10 (c) 13 (d) 12

4. The remainder when $(32^{32})^{32}$ is divided by 7, is
(a) 2 (b) 4 (c) 6 (d) 5

5. Seven different letters are given. Then the number of ways in which words of 5 letters can be formed such that atleast one of the letters is repeated, is
(a) ${}^7P_5$ (b) 14287 (c) 5^7 (d) 7^5

6. The sum of the coefficients in the expansion of $(1-x)^{10}$ is
(a) 0 (b) 1 (c) 10^2 (d) 1024

7. If ${}^{n-1}C_3+{}^{n-1}C_4>{}^nC_3$, then
(a) $n>5$ (b) $n>6$
(c) $n>7$ (d) None of these

8. The coefficient of x^n in the binomial expansion of $(1-x)^{-2}$ is
(a) $\frac{2^n}{2!}$ (b) $n+1$ (c) n (d) $2n$

9. The triplet (x, y, z) is chosen from the set $\{1, 2, 3, ..., n\}$, such that $x \le y < z$. The number of such triplets is
(a) n^3 (b) ${}^nC_3$
(c) ${}^nC_2$ (d) ${}^nC_3+{}^nC_2$

10. In how many ways a committee consisting of 3 men and 2 women, can be chosen from 7 men and 5 women?
(a) 45 (b) 350 (c) 4200 (d) 230

11. Find n, if ${}^nP_4 : {}^nP_5 = 1:2$.

12. Find the number of terms in the expansion of $(\sqrt{x}+\sqrt{y})^{10}+(\sqrt{x}-\sqrt{y})^{10}$.

13. From a class of 36 students, 3 students are to be selected to take part in a competition. How many such selections can be made?

14. Evaluate : ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$

15. Write the general term in the expansion of $(x^2 - y)^6$.

16. How many permutations of the letters of the word 'APPLE' are there?

17. Expand : $(a + b)^6 - (a - b)^6$

18. If ${}^{10}C_x = {}^{10}C_{x+4}$, then find the value of x.

19. Find a positive value of m for which the coefficient of x^2 in the expansion of $(1 + x)^m$ is 6.

20. If ${}^nP_r = 720$ and ${}^nC_r = 120$, then find the value of r.

SECTION - B

21. In how many ways 5 rings of different types can be worn in 4 fingers?

22. Which is larger $(1.01)^{1000000}$ or 10000?

23. Find the number of ways of dividing 9 different balls into 5 groups, four of them containing 2 balls each and the fifth only one ball.

OR

Prove that ${}^nC_r \times {}^rC_s = {}^nC_s \times {}^{n-s}C_{r-s}$.

24. Prove that ${}^nP_r = {}^{n-1}P_r + r \cdot {}^{n-1}P_{r-1}$.

25. How many seven digit numbers can be formed using only three digits 3, 5, 8; the digit 5 occurring exactly twice in each number?

26. Find the value of α, for which the coefficient of the middle terms in the expansions of $(1 + \alpha x)^4$ and $(1 - \alpha x)^6$ are equal.

OR

In the binomial expansion of $(\sqrt[3]{3} + \sqrt{2})^5$, find the term which does not contain irrational expression.

SECTION - C

27. How many lawn-tennis mixed doubles games can be arranged from 7 married couples if no husband and wife pair is included in the same game?

28. If the coefficients of three consecutive terms in the expansion of $(1 + a)^n$ are in the ratio 1 : 7 : 42, then find the value of n.

OR

If the coefficient of x^r in the expansion of $(1 - x)^{2n-1}$ is denoted by a_r, then prove that $a_{r-1} + a_{2n-r} = 0$.

29. Find the number of ways in which 5 boys and 5 girls be seated in a row so that

(i) no two girls may sit together.

(ii) all the girls sit together and all the boys sit together.

(iii) all the girls are never together.

30. In how many ways can a Cricket team of eleven players be chosen out of a batch of 15 players, if

(i) there is no restriction on the selection?

(ii) a particular player is always chosen?

(iii) a particular player is never chosen?

OR

A question paper contains 12 questions, divided into three parts. Part A contains 6 questions while B and C contains 3 questions each. A candidate is required to attempt 6 questions selecting at least two from part A and at least one from each of part B and part C. In how many ways can the candidate select 6 questions?

31. Find the value of k, so that the term independent of x in $\left(\sqrt{x} + \dfrac{k}{x^2}\right)^{10}$ is 405.

32. If all the letters of the word 'MOTHER' are written in all possible orders and the words so formed are arranged as in a dictionary order, then find the position of the word 'MOTHER'.

SECTION - D

33. For what value of x, the ninth term in the expansion of $\left\{3^{\log_3 \sqrt{25^{x-1}+7}} + 3^{(-1/8)\log_3 (5^{x-1}+1)}\right\}^{10}$ is equal to 180?

34. Find the number of rectangles formed on a chess board. How many of these are squares?

OR

From a class of 12 boys and 10 girls, 10 students are to be chosen for a competition, including atleast 4 boys and 4 girls. The two girls who won the prizes last year should be included. In how many ways can the selection be made?

35. Find the number of integral terms in the expansion of $(5^{1/2} + 7^{1/8})^{1024}$.

36. Find the number of 4-digit numbers greater than 4000 whose sum of digits is odd.

OR

To fill 12 vacancies, there are 25 candidates of which 5 are from scheduled castes. If 3 vacancies are reserved for scheduled caste candidates while the rest are open to all, find the number of ways in which the selections can be made.

SOLUTIONS

1. (b) : We have, $\frac{1}{8!}+\frac{1}{9\times 8!}=\frac{x}{10\times 9\times 8!}$

$\therefore\quad 1+\frac{1}{9}=\frac{x}{10\times 9} \Rightarrow \frac{10}{9}=\frac{x}{10\times 9} \quad\Rightarrow\quad x = 100$

2. (b) : Here, $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$

$\Rightarrow\ z=2\left[{}^5C_0\left(\frac{\sqrt{3}}{2}\right)^5+{}^5C_2\left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{i}{2}\right)^2+{}^5C_4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{i}{2}\right)^4\right]$

Hence, $\text{Im}(z) = 0$. $[\because\ i^2 = -1 \text{ and } i^4 = 1]$

3. (b) : Given that, $\frac{{}^nP_4}{{}^{n-1}P_4}=\frac{5}{3}$

$\Rightarrow\ 3n(n-1)(n-2)(n-3) = 5(n-1)(n-2)(n-3)(n-4)$

$\Rightarrow\ 3n = 5(n-4)\quad [\because\ (n-1)(n-2)(n-3) \neq 0, n > 4]$

$\Rightarrow\ n = 10$

4. (b) : $(32^{32})^{32} = 32^{1024} = (28 + 4)^{1024}$
$= 4 +$ a multiple of $28 = 4 +$ a multiple of 7.

5. (b) : Required number of ways $= 7^5 - {}^7P_5 = 14287$.

6. (a) : $(1-x)^{10} = 1 - {}^{10}C_1\, x + {}^{10}C_2\, x^2 - {}^{10}C_3\, x^3 + \ldots$
$\ldots + {}^{10}C_{10}\, x^{10}$

$\therefore$ Sum of the coefficients $= 1 - {}^{10}C_1 + {}^{10}C_2 - {}^{10}C_3 + \ldots$
$+ {}^{10}C_{10} = 0$

7. (c) : We have, ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$

$\Rightarrow\ {}^nC_4 > {}^nC_3 \Rightarrow \frac{n!}{(n-4)!4!}>\frac{n!}{(n-3)!\,3!}$

$\Rightarrow\ \frac{1}{4}>\frac{1}{n-3} \Rightarrow n-3>4 \Rightarrow n>7.$

8. (b) : $\because\ (1-x)^{-2} = 1 + 2x + 3x^2 + \ldots + (n+1)x^n + \ldots$

$\therefore$ Coefficient of $x^n = n + 1$.

9. (d) : Number of selections when $x < y < z$ is ${}^nC_3$.
Number of selections when $x = y < z$ is ${}^nC_2$.

$\therefore$ Required number $= {}^nC_3 + {}^nC_2$

10. (b) : Out of 7 men, 3 men can be chosen in ${}^7C_3$ ways and out of 5 women, 2 women can be chosen in ${}^5C_2$ ways. Hence, the committee can be chosen in ${}^7C_3 \times {}^5C_2 = 350$ ways.

11. Given, ${}^nP_4 : {}^nP_5 = 1 : 2$

$\Rightarrow\ \frac{n(n-1)(n-2)(n-3)}{n(n-1)(n-2)(n-3)(n-4)}=\frac{1}{2}$

$\Rightarrow\ n - 4 = 2 \quad\Rightarrow\quad n = 6.$

12. Since n is even, so the expansion

$(\sqrt{x}+\sqrt{y})^{10}+(\sqrt{x}-\sqrt{y})^{10}$ has $\left(\frac{10}{2}+1\right)$ = 6 terms.

13. The required number of selections $= {}^{36}C_3$

$=\frac{36\times 35\times 34}{1\times 2\times 3}=7140$

14. We have, ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$

$= ({}^{15}C_8 + {}^{15}C_9) - ({}^{15}C_6 + {}^{15}C_7) = {}^{16}C_9 - {}^{16}C_7 = 0$

$[\because\ {}^{16}C_9 = {}^{16}C_7]$

15. The general term of $(x^2 - y)^6 = [x^2 + (-y)]^6$ is given by $T_{r+1} = {}^6C_r\, (x^2)^{6-r}\, (-y)^r = (-1)^r\, {}^6C_r\, x^{12-2r}\, y^r$

16. There are 5 letters, two of which are of the same kind. The others are each of its own kind.

So, the required number of permutations $= \frac{5!}{2!1!1!1!}$

$= 5 \times 4 \times 3 = 60.$

17. We have, $(a + b)^6 - (a - b)^6$

$= ({}^6C_0\, a^6 + {}^6C_1\, a^5b + {}^6C_2\, a^4b^2 + {}^6C_3\, a^3b^3 + {}^6C_4\, a^2b^4$
$+ {}^6C_5\, ab^5 + {}^6C_6\, b^6) - ({}^6C_0\, a^6 - {}^6C_1\, a^5b + {}^6C_2\, a^4b^2 -$
${}^6C_3\, a^3b^3 + {}^6C_4\, a^2b^4 - {}^6C_5\, ab^5 + {}^6C_6\, b^6)$

$= 2[{}^6C_1\, a^5b + {}^6C_3\, a^3b^3 + {}^6C_5\, ab^5]$

$= 2[6a^5b + 20a^3 b^3 + 6ab^5] = 4ab\,[3a^4 + 10a^2b^2 + 3b^4]$

18. We have, ${}^{10}C_x = {}^{10}C_{x+4}$

$\Rightarrow\ x + x + 4 = 10 \quad\Rightarrow\quad 2x = 6 \quad\Rightarrow\quad x = 3.$

19. Coefficient of x^2 in $(1 + x)^m$ is ${}^mC_2$.

$\therefore\ {}^mC_2 = 6$ [Given] $\Rightarrow \frac{m(m-1)}{2}=6$

$\Rightarrow\ m^2 - m - 12 = 0 \quad\Rightarrow\quad (m-4)(m+3) = 0$

$\Rightarrow\ m = 4, -3$ but m is +ve $\Rightarrow\ m = 4$.

20. Since we know that ${}^nC_r=\frac{{}^nP_r}{r!}$

$\therefore\ 120=\frac{720}{r!}\quad [\because\ {}^nC_r = 120 \text{ and } {}^nP_r = 720 \text{ (Given)}]$

$\Rightarrow\ r! = 6 \quad\Rightarrow\quad r! = 3! \quad\Rightarrow\quad r = 3.$

21. The first ring can be worn in any of the 4 fingers. So, there are 4 ways of wearing it. Similarly, each one of the other rings can be worn in 4 ways.
Hence, the required number of ways
$= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

22. $(1.01)^{1000000} = (1 + 0.01)^{1000000}$
$= {}^{1000000}C_0 + {}^{1000000}C_1\, (0.01)$ + other positive terms
$= 1 + 1000000 \times 0.01$ + other positive terms
$= 1 + 10000$ + other positive terms > 10000
Hence, the larger number is $(1.01)^{1000000}$.

23. Required number of ways

$=\frac{9!}{(2!)^4\,1!4!}=\frac{9\times 8\times 7\times 6\times 5\times 4!}{2^4\times 4!}=\frac{9\times 8\times 7\times 6\times 5}{16}=945.$

OR

$$\text{L.H.S.} = {}^nC_r \times {}^rC_s = \frac{n!}{r!(n-r)!}\cdot\frac{r!}{s!(r-s)!}$$

$$= \frac{n!}{(n-r)!(r-s)!s!} \qquad \text{...(i)}$$

$$\text{R.H.S.} = {}^nC_s \times {}^{n-s}C_{r-s} = \frac{n!}{s!(n-s)!}\cdot\frac{(n-s)!}{(r-s)!(n-r)!}$$

$$= \frac{n!}{s!(r-s)!(n-r)!} \qquad \text{...(ii)}$$

$\therefore$ From (i) and (ii), L.H.S. = R.H.S.

24. R.H.S. = ${}^{n-1}P_r + r\cdot{}^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-1-r)!} + r\cdot\frac{(n-1)!}{(n-1-(r-1))!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r\cdot\frac{(n-1)!}{(n-r)!} = \frac{(n-1)!}{(n-r-1)!} + \frac{r(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left(1+\frac{r}{n-r}\right) = \frac{(n-1)!}{(n-r-1)!}\cdot\frac{n-r+r}{(n-r)}$$

$$= \frac{n(n-1)!}{(n-r)(n-r-1)!} = \frac{(n)!}{(n-r)!} = {}^nP_r = \text{L.H.S.}$$

25. Any two of the seven places can be chosen in ${}^7C_2$ ways and in these places, the digit 5 is to be put. The remaining five places are filled with either 3 or 8 in 2^5 ways.

$\therefore$ The required numbers = ${}^7C_2 \times 2^5 = 21 \times 32 = 672$.

26. In the expansion of $(1 + \alpha x)^4$, middle term $= {}^4C_2\,(\alpha x)^2 = 6\alpha^2 x^2$

In the expansion of $(1 - \alpha x)^6$, middle term $= {}^6C_3(-\alpha x)^3 = -20\alpha^3 x^3$

Since it is given that coefficient of the middle term in $(1 + \alpha x)^4$ = Coefficient of the middle term in $(1 - \alpha x)^6$

$\therefore\ 6\alpha^2 = -20\alpha^3 \Rightarrow \alpha = 0, -3/10.$

OR

We have,

$$(\sqrt[3]{3}+\sqrt{2})^5 = {}^5C_0(\sqrt[3]{3})^5 + {}^5C_1(\sqrt[3]{3})^4\sqrt{2} + {}^5C_2(\sqrt[3]{3})^3(\sqrt{2})^2 + {}^5C_3(\sqrt[3]{3})^2(\sqrt{2})^3 + {}^5C_4(\sqrt[3]{3})^1(\sqrt{2})^4 + {}^5C_5(\sqrt{2})^5$$

$$= (3)^{5/3} + 5(3)^{4/3}\sqrt{2} + 10\cdot3\cdot2 + 10\cdot(3)^{2/3}\cdot2\sqrt{2} + 5\cdot\sqrt[3]{3}\cdot4 + 4\sqrt{2}$$

$\therefore$ The term which does not contain irrational expression is the third term, *i.e.*, $10 \times 3 \times 2 = 60$.

27. In order to arrange the lawn tennis mixed doubles game, first we select two men from 7 men in ${}^7C_2$ ways. Then exclude the wives of the men already selected and select two women out of the remaining 5 women in ${}^5C_2$ ways.

Now, arrange the 4 selected persons in two teams. If the selected men are M_1 and M_2 and the selected women are W_1 and W_2, the teams can be formed in 2 ways *i.e.*, M_1W_1 against M_2W_2 or M_1W_2 against M_2W_1.

Hence, the number of ways of arranging the game $= {}^7C_2 \times {}^5C_2 \times 2 = 21 \times 10 \times 2 = 420$.

28. Let the three consecutive terms in the expansion of $(1 + a)^n$ be r^{th}, $(r + 1)^{\text{th}}$ and $(r + 2)^{\text{th}}$ terms respectively.

Now in the expansion of $(1 + a)^n$, coefficient of r^{th} term $= {}^nC_{r-1}$, coefficient of $(r + 1)^{\text{th}}$ term $= {}^nC_r$ and coefficient of $(r + 2)^{\text{th}}$ term $= {}^nC_{r+1}$.

Given, ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 1 : 7 : 42$

$$\therefore\ \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{7} \Rightarrow \frac{n!}{(r-1)!(n-r+1)!}\cdot\frac{r!(n-r)!}{n!} = \frac{1}{7}$$

$$\Rightarrow \frac{r}{n-r+1} = \frac{1}{7} \Rightarrow 7r = n - r + 1$$

$$\Rightarrow n - 8r = -1 \qquad \text{...(i)}$$

$$\text{and } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{7}{42} \Rightarrow \frac{n!}{r!(n-r)!}\cdot\frac{(r+1)!(n-r-1)!}{n!} = \frac{1}{6}$$

$$\Rightarrow \frac{r+1}{n-r} = \frac{1}{6} \Rightarrow 6r + 6 = n - r \Rightarrow n - 7r = 6 \qquad \text{...(ii)}$$

Solving (i) and (ii), we have $r = 7$ and $n = 55$.

OR

In the expansion of $(1 - x)^{2n-1}$, $T_{r+1} = {}^{2n-1}C_r(-x)^r$

Now, coefficient of $x^r = (-1)^r\cdot{}^{2n-1}C_r = a_r$ [Given]

$\Rightarrow a_{r-1} = (-1)^{r-1}\cdot{}^{2n-1}C_{r-1}$ and $a_{2n-r} = (-1)^{2n-r}\cdot{}^{2n-1}C_{2n-r}$

$\therefore\ a_{r-1} + a_{2n-r} = (-1)^{r-1}\cdot{}^{2n-1}C_{r-1} + (-1)^{2n-r}\cdot{}^{2n-1}C_{2n-r}$

$= (-1)^{r-1}\cdot{}^{2n-1}C_{(2n-1)-(r-1)} + (-1)^{2n}(-1)^{-r}\cdot{}^{2n-1}C_{2n-r}$

[$\because\ {}^nC_r = {}^nC_{n-r}$]

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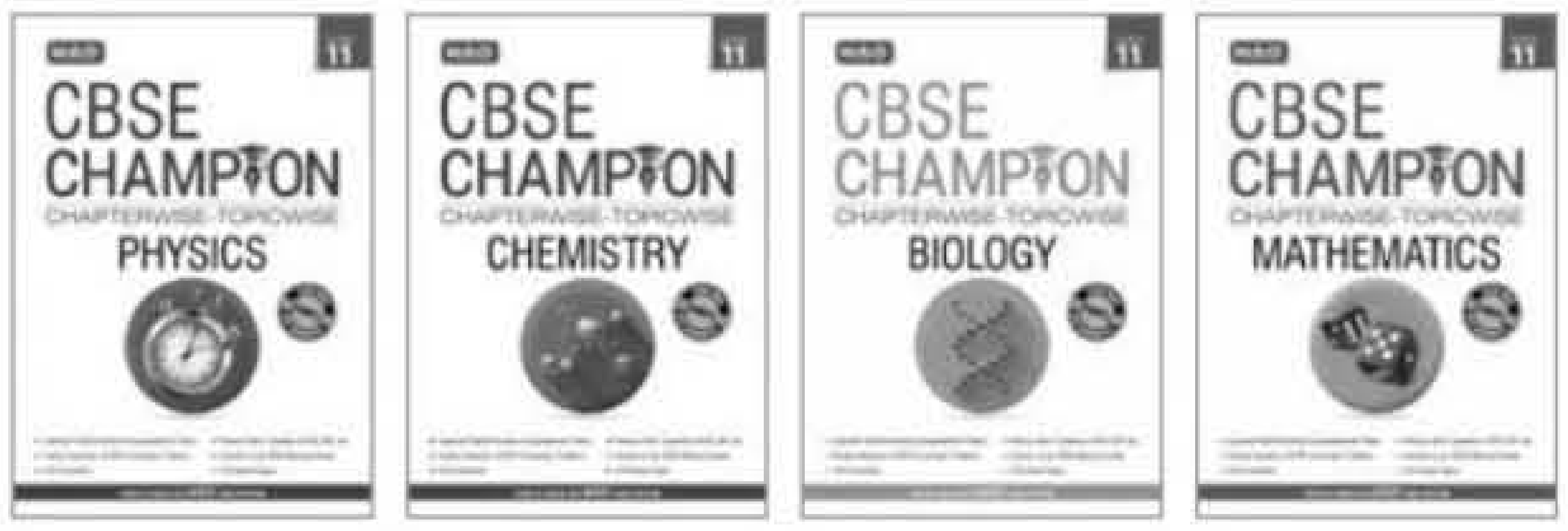

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- Important Facts/Formulae & Comprehensive Theory
- Practice Questions, NCERT & Exemplar Problems
- HOTS Questions
- Previous Years' Questions of KVS, NCT, etc.
- Answers as per CBSE Marking Scheme
- 10 Practice Papers

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$$= (-1)^{r-1} \cdot {}^{2n-1}C_{2n-r} + \frac{1}{(-1)^r}\ {}^{2n-1}C_{2n-r} \quad [\because\ (-1)^{2n} = 1]$$

$$= \left[(-1)^{r-1} + \frac{1}{(-1)^r}\right] {}^{2n-1}C_{2n-r} = \frac{(-1)^{2r-1}+1}{(-1)^r} \cdot {}^{2n-1}C_{2n-r}$$

$$= 0 \qquad \left[\because (-1)^{2r-1} = \frac{(-1)^{2r}}{(-1)^1} = \frac{1}{-1} = -1\right]$$

29. (i) 5 boys can be seated in row in $^5P_5$ = 5! ways. Now, in the 6 gaps 5 girls can be arranged in $^6P_5$ ways. Hence, the number of ways in which no two girls sit together = 5! × $^6P_5$ = 5! × 6!.

(ii) Two groups of girls and boys can be arranged in 2! ways. 5 girls can be arranged among themselves in 5! ways. Similarly, 5 boys can be arranged among themselves in 5! ways. Hence, by fundamental principle of counting, the total number of required seating arrangements = $2!(5! \times 5!) = 2(5!)^2$.

(iii) Total number of ways in which all the girls are never together = Total number of arrangements – Total number of arrangement in which all the girls are always together = 10! – (5! × 6!).

30. (i) The total number of ways of selecting 11 players out of 15 is $^{15}C_{11} = \frac{15\times14\times13\times12}{4\times3\times2\times1} = 1365$

(ii) If a particular player is always chosen, then 10 players are selected out of the remaining 14 players.

∴ Required number of ways $= {}^{14}C_{10} = \frac{14\times13\times12\times11}{4\times3\times2\times1}$

= 1001

(iii) If a particular player is never chosen, then 11 players are selected out of the remaining 14 players.

∴ Required number of ways $= {}^{14}C_{11} = \frac{14\times13\times12}{3\times2\times1}$

= 364

OR

Refer to answer 93, Page no. 155 of MTG CBSE Champion Mathematics, Class-11

31. In the expansion of $\left(\sqrt{x}+\frac{k}{x^2}\right)^{10}$, r^{th} term is given by

$$T_r = {}^{10}C_{r-1}(\sqrt{x})^{10-r+1}\left(\frac{k}{x^2}\right)^{r-1} = {}^{10}C_{r-1}(\sqrt{x})^{11-r}\frac{k^{r-1}}{x^{2r-2}}$$

$$= {}^{10}C_{r-1}\,x^{\frac{11-r}{2}} \times \frac{k^{r-1}}{x^{2r-2}} = {}^{10}C_{r-1}\,x^{\frac{11-r}{2}-2r+2}\,k^{r-1}$$

$$\therefore\ T_r = {}^{10}C_{r-1}x^{\frac{15-5r}{2}}k^{r-1} \qquad \text{...(i)}$$

Since, r^{th} term is independent of x.

$$\therefore\ \frac{15-5r}{2} = 0 \Rightarrow r = 3$$

Putting $r = 3$ in (i), we get $T_3 = {}^{10}C_2\,k^2$

Now, $^{10}C_2\,k^2 = 405$ [Given]

$\Rightarrow\ 45k^2 = 405 \Rightarrow k^2 = 9 \therefore k = \pm 3.$

32. In a dictionary, the words at each stage are arranged in a alphabetical order. The letters of the word 'MOTHER' arranged alphabetically, are E, H, M, O, R, T.

Number of words beginning with E = 5! = 120

Number of words beginning with H = 5! = 120

Number of words beginning with M = 5! = 120, but one of these words is MOTHER.

Number of words beginning with ME = 4! = 24

Number of words beginning with MH = 4! = 24

Number of words beginning with MO = 4! = 24, but one of these words is MOTHER.

Number of words beginning with MOE = 3! = 6

Number of words beginning with MOH = 3! = 6

Number of words beginning with MOR = 3! = 6

Number of words beginning with MOTE = 2! = 2

The total number of words till now = 120 + 120 + 24 + 24 + 6 + 6 + 6 + 2 = 308

The next word is 'MOTHER'.

∴ The position of the word MOTHER = 309.

33. Consider, $\left\{3^{\log_3\sqrt{25^{x-1}+7}} + 3^{(-1/8)\log_3(5^{x-1}+1)}\right\}^{10}$

$$= \left\{\sqrt{25^{x-1}+7} + (5^{x-1}+1)^{(-1/8)}\right\}^{10}$$

Let T_9 be the 9th term in the above expansion.

Then, $T_9 = 180$ [Given]

$$\Rightarrow\ {}^{10}C_8\left\{\sqrt{25^{x-1}+7}\right\}^{10-8}\left\{(5^{x-1}+1)^{(-1/8)}\right\}^8 = 180$$

$$\Rightarrow\ {}^{10}C_8\,(25^{x-1}+7)\,(5^{x-1}+1)^{-1} = 180$$

$$\Rightarrow\ \frac{45(25^{x-1}+7)}{5^{x-1}+1} = 180 \ \Rightarrow\ \frac{25^{x-1}+7}{5^{x-1}+1} = 4$$

$$\Rightarrow\ \frac{y^2+7}{y+1} = 4, \text{ where } y = 5^{x-1}$$

$\Rightarrow\ y^2 - 4y + 3 = 0 \ \Rightarrow\ (y-3)(y-1) = 0$

$\Rightarrow\ y = 3, 1 \ \Rightarrow\ 5^{x-1} = 3$ or $5^{x-1} = 1$

$\Rightarrow\ 5^x = 15$ or $5^x = 5 \ \Rightarrow\ x = \log_5 15$ or $x = 1$.

34. A chess board has 64 squares of sides say 1 unit. It has 9 horizontal lines and 9 vertical lines.

∴ The number of rectangles so formed on the chess board = $^9C_2 \times {}^9C_2$ = 36 × 36 = 1296.

A chess board has squares of different size.

Number of 1 × 1 squares = 64

Number of 2 × 2 squares = 49
Number of 3 × 3 squares = 36
Number of 4 × 4 squares = 25
Number of 5 × 5 squares = 16
Number of 6 × 6 squares = 9
Number of 7 × 7 squares = 4
Number of 8 × 8 squares = 1

$\therefore$ The total number of squares formed on the chess board = 64 + 49 + 36 + 25 + 16 + 9 + 4 + 1

$= 1^2 + 2^2 + 3^2 + \dots + 8^2$

$$= \frac{8(8+1)(2\times8+1)}{6} = \frac{8\times9\times17}{6} = 204.$$

OR

Refer to answer 94, Page no. 155 of MTG CBSE Champion Mathematics, Class-11

35. The general term T_{r+1} in the expansion of $(5^{1/2} + 7^{1/8})^{1024}$ is given by

$$T_{r+1} = {}^{1024}C_r (5^{1/2})^{1024-r} (7^{1/8})^r = {}^{1024}C_r 5^{512-\frac{r}{2}} 7^{r/8}$$

$$= \{{}^{1024}C_r 5^{512-r}\} \times 5^{r/2} \times 7^{r/8} = \{{}^{1024}C_r 5^{512-r}\} \times (5^4 \times 7)^{r/8}$$

Clearly, T_{r+1} will be an integer, iff $r/8$ is an integer such that $0 \le r \le 1024$

$\Rightarrow$ r is a multiple of 8, satisfying $0 \le r \le 1024$

$\Rightarrow$ $r = 0, 8, 16, 24, \dots, 1024 \Rightarrow r$ can assume 129 values.

Hence, there are 129 integral terms in the expansion of $(5^{1/2} + 7^{1/8})^{1024}$.

36. We have to form 4-digit numbers greater than 4000 whose sum of digits is odd. So, we can use any digit any number of times.

Since, the required numbers are greater than 4000, the thousand's place can be filled in 6 ways (be any of digits 4, 5, 6, 7, 8, 9).

Hundred's place can be filled in 10 ways and ten's place can also be filled in 10 ways.

Thus, the first 3 places (thousand's, hundred's and ten's) can be filled in 6 × 10 × 10 ways = 600 ways.

Now, we want that the sum of all digits of the number formed should be odd. So, two cases arise:

Case I : If the sum of first three places is odd, then fill the unit's place by any even digit (0, 2, 4, 6, 8).

Case II : If the sum of first three places is even, then fill the unit's place by any odd digit (1, 3, 5, 7, 9).

In either case, there are 5 ways to fill the unit's place.

$\therefore$ The required numbers formed = 600 × 5 = 3000.

OR

There are 25 candidates of which 5 are from scheduled castes, so the other candidates are 20.

Now, 12 vacancies are to filled out of which 3 vacancies are reserved for scheduled caste candidates and other vacancies are open to all.

The different mutually exclusive possibilities are :

(i) 3 from scheduled caste candidates and 9 from others.

Number of ways of selections $= {}^5C_3 \times {}^{20}C_9 = {}^5C_2 \times {}^{20}C_9$
$= 10 \times {}^{20}C_9$.

(ii) 4 from scheduled caste candidates and 8 from others.

Number of ways of selections $= {}^5C_4 \times {}^{20}C_8 = {}^5C_1 \times {}^{20}C_8$
$= 5 \times {}^{20}C_8$.

(iii) 5 from scheduled caste candidates and 7 from others.

Number of ways of selections $= {}^5C_5 \times {}^{20}C_7 = 1 \times {}^{20}C_7$
$= {}^{20}C_7$.

$\therefore$ The total number of ways of selections
$= 10 \times {}^{20}C_9 + 5 \times {}^{20}C_8 + {}^{20}C_7$.

◆

SAMURAI
SUDOKU

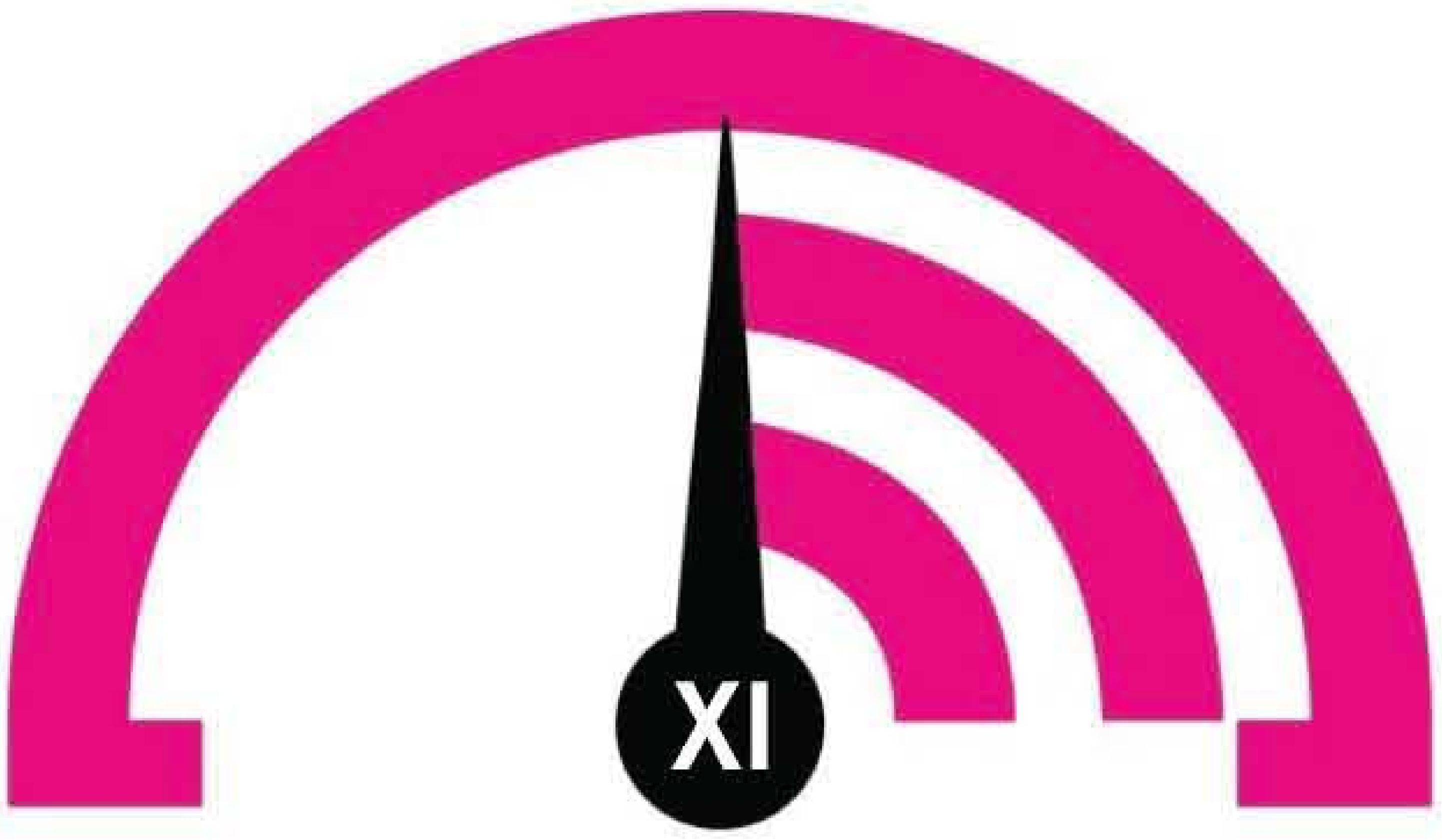
Samurai Sudoku puzzle consists of five overlapping sudoku grids. The standard sudoku rules apply to each 9 × 9 grid. Place digits from 1 to 9 in each empty cell. Every row, every column and every 3 × 3 box should contain one of each digit.

The puzzle has a unique answer.

	7	9					6	5				1	7							3
4	5				7								9					8		
				8			1						2							
	8		7	2	5	3								1			6		3	
		5		6												1			2	8
2			1										3		2		7			4
6							7	2		8					7		1	5		
7		2			3	6	4				3									6
								1	7					2	5	4		9	1	
									1		2		5							
						4							9	1						
								6		4			3							
1	5					7				9				4		6	7		8	
	9			6	4		2					6							7	4
		3	5				9		2		6				4				6	2
					8	3								1	3				9	
			1	5	9	2								6		8				
5								1				4	3		9	1				
8				2	3							5	1							
2			7		6								4							6
	3		9			6		2						7	8		2	3		

Readers can send their responses at editor@mtg.in or post us with complete address. Winners' name with their valuable feedback will be published in next issue.

MONTHLY TEST DRIVE



This specially designed column enables students to self analyse their extent of understanding of specified chapters. Give yourself four marks for correct answer and deduct one mark for wrong answer. Self check table given at the end will help you to check your readiness.

Total Marks : 80

Series 3 : Trigonometric Functions

Time Taken : 60 Min.

Only One Option Correct Type

1. If $\sin 3\theta = \sin\theta$, then how many solutions exist such that $-2\pi < \theta < 2\pi$?

(a) 8 (b) 9
(c) 5 (d) None of these

2. If $\tan^2\alpha\ \tan^2\beta + \tan^2\beta\ \tan^2\gamma + \tan^2\gamma\ \tan^2\alpha + 2\tan^2\alpha\ \tan^2\beta\ \tan^2\gamma = 1$, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is

(a) 0 (b) -1
(c) 1 (d) None of these

3. In a ΔABC, $A : B : C = 3 : 5 : 4$, then $a + b + c\sqrt{2}$ is equal to

(a) $2b$ (b) $2c$
(c) $3b$ (d) $3a$

4. The solution of the equation $\log_{\cos x}\sin x + \log_{\sin x}\cos x = 2$ is given by

(a) $x = 2n\pi + \dfrac{\pi}{4}, n \in I$ (b) $x = n\pi + \dfrac{\pi}{2}, n \in I$

(c) $x = n\pi + \dfrac{\pi}{8}, n \in I$ (d) None of these

5. In any triangle, with usual notations the minimum value of $r_1r_2r_3/r^3$ is equal to

(a) 1 (b) 9
(c) 27 (d) None of these

6. If $\cos^3 x \sin 2x = \sum_{r=0}^{n} a_r \sin(rx) \forall\ x \in R$, then

(a) $n = 5, a_1 = 1/2$ (b) $n = 5, a_1 = 1/4$
(c) $n = 5, a_2 = 1/8$ (d) $n = 5, a_2 = 1/4$

One or More Than One Option(s) Correct Type

7. If $\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x = 7$ and $\sin 2x = a - b\sqrt{7}$, then

(a) $a = 8$ (b) $b = 22$ (c) $a = 22$ (d) $b = 8$

8. If $\cos(\theta - \alpha)$, $\cos\theta$, $\cos(\theta + \alpha)$ are in H.P., then $\cos\theta \sec\dfrac{\alpha}{2}$ is equal to

(a) -1 (b) $-\sqrt{2}$ (c) $\sqrt{2}$ (d) 2

……

DO YOU KNOW ?

Amazing truth

Letters 'a', 'b', 'c' & 'd' do not appear anywhere in the spellings of 1 to 99

(Letter 'd' comes for the first time in Hundred)

Letters 'a', 'b' & 'c' do not appear anywhere in the spellings of 1 to 999

(Letter 'a' comes for the first time in Thousand)

Letters 'b' & 'c' do not appear anywhere in the spellings of 1 to 999,999,999

(Letter 'b' comes for the first time in Billion)

And

Letter 'c' does not appear anywhere in the spellings of entire International Counting

9. $\sqrt{\cos 2x}+\sqrt{(1+\sin 2x)}=2\sqrt{(\sin x+\cos x)}$, if

(a) $\sin x + \cos x = 0$

(b) $x = 2n\pi,\ n \in I$

(c) $x=n\pi-\frac{\pi}{4},\ n\in I$

(d) $x=2n\pi \pm \cos^{-1}\left(-\frac{1}{5}\right),\ n\in I$

10. Let $f_n(\theta)=\tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)\ (1 + \sec 4\theta)\ldots$ $(1 + \sec 2^n\theta)$, then

(a) $f_2\left(\frac{\pi}{16}\right)=1$ (b) $f_3\left(\frac{\pi}{32}\right)=1$

(c) $f_4\left(\frac{\pi}{64}\right)=1$ (d) $f_5\left(\frac{\pi}{128}\right)=1$

11. If $0\le x\le 2\pi$, then $2^{\mathrm{cosec}^2 x}\sqrt{\frac{1}{2}y^2-y+1}\le\sqrt{2}$

(a) is satisfied by exactly one value of y

(b) is satisfied by exactly two values of x

(c) is satisfied by x for which $\sin x = 0$

(d) None of these

12. If A is the area and $2s$ is the sum of the sides of a triangle, then

(a) $A\le\frac{s^2}{4}$ (b) $A\le\frac{s^2}{3\sqrt{3}}$

(c) $A<\frac{s^2}{\sqrt{3}}$ (d) None of these

13. In a ΔABC,

(a) $\sin A \sin B \sin C \le \frac{3\sqrt{3}}{8}$

(b) $\sin^2 A + \sin^2 B + \sin^2 C \le \frac{9}{4}$

(c) $\sin A \sin B \sin C$ is always positive

(d) $\sin^2 A + \sin^2 B \le 1 + \cos C$

Comprehension Type

Consider the cubic equation $x^3 - (1 + \cos\theta + \sin\theta)x^2 + (\cos\theta\sin\theta + \cos\theta + \sin\theta)x - \sin\theta\cos\theta = 0$, whose roots are x_1, x_2 and x_3.

14. The value of $x_1^2 + x_2^2 + x_3^2$ equals

(a) 1 (b) 2

(c) $2\cos\theta$ (d) $\sin\theta(\sin\theta + \cos\theta)$

15. The number of values of θ in $[0, 2\pi]$ for which at least two roots are equal, are

(a) 3 (b) 4 (c) 5 (d) 6

Matrix Match Type

16. Match the following :

Column-I		Column-II	
P.	The value of $\left(\frac{4+\sec 20^\circ}{\mathrm{cosec}\,20^\circ}\right)^2$, is	1.	1
Q.	The minimum value of $\frac{1+\cos 2x+8\sin 2x}{2\sin 2x}$, $x\in(0, \pi/2)$, is	2.	2
R.	The value of $\frac{8\sin 40^\circ\cdot\sin 50^\circ\cdot\tan 10^\circ}{\cos 80^\circ}$ is	3.	3
S.	If $\frac{\cos 5A}{\cos A}+\frac{\sin 5A}{\sin A}=a+b\cos 4A$, then a^2/b is	4.	4

	P	Q	R	S		P	Q	R	S
(a)	1	3	2	4	(b)	3	2	4	1
(c)	4	1	3	2	(d)	2	4	1	3

Numerical Answer Type

17. The sum of the roots of equation $\cos 4x + 6 = 7\cos 2x$ over the interval $[0, 314]$ is $\lambda\pi$, then the numerical quantity λ must be

18. If $\frac{1}{\cos 290^\circ}+\frac{1}{\sqrt{3}\sin 250^\circ}=\lambda$, then the value of $9\lambda^4 + 81\lambda^2 + 97$ must be

19. The value of $f(x) = x^4 + 4x^3 + 2x^2 - 4x + 7$, when $x=\cot\frac{11\pi}{8}$ is

20. Consider a ΔABC, in which the sides are $a = (n + 1)$, $b = (n + 2)$, $c = n$ with $\tan C = 4/3$, then the value of $\Delta/12$ is

◆◆

Keys are published in this issue. Search now! ☺

SELF CHECK

No. of questions attempted ......

No. of questions correct ......

Marks scored in percentage ......

Check your score! If your score is

> 90%	EXCELLENT WORK !	You are well prepared to take the challenge of final exam.
90-75%	GOOD WORK !	You can score good in the final exam.
74-60%	SATISFACTORY !	You need to score more next time.
< 60%	NOT SATISFACTORY!	Revise thoroughly and strengthen your concepts.

A Legendary Mathematician in Madras

A charming story of the Hungarian genius Paul Erdős and how he encouraged young students

On a winter's day in January 1975, two men walked down the sandy stretch of Marina Beach in Chennai. It was early afternoon, but there was a breeze blowing, and they had the place pretty much to themselves. One was a final year B.Sc. student from Ramakrishna Mission Vivekananda College; the other an older, frail-looking foreigner. Initially, the older man asked about some landmarks on that stretch, including the impressive Indo-Saracenic building that houses the University of Madras, but after that the conversation drifted to advanced mathematics. Clearly, this man was no ordinary tourist.

Paul Erdős, the legendary Hungarian mathematician, was on his first trip to Chennai, or Madras, as it was called then. At 21, he had earned his Ph.D. from the University of Budapest. This was in 1934. In the next six decades, he would go on to publish over 1,500 papers, an unsurpassed record. He made fundamental contributions to certain branches of mathematics — number theory, in particular — and pioneered discrete mathematics, the foundation of computer science. A bachelor, he had no permanent job or home address.

In the pre-Internet era, he connected researchers across the globe who might otherwise have toiled away on problems on their own, making little headway. His life's mission was to discover and nurture young mathematicians.

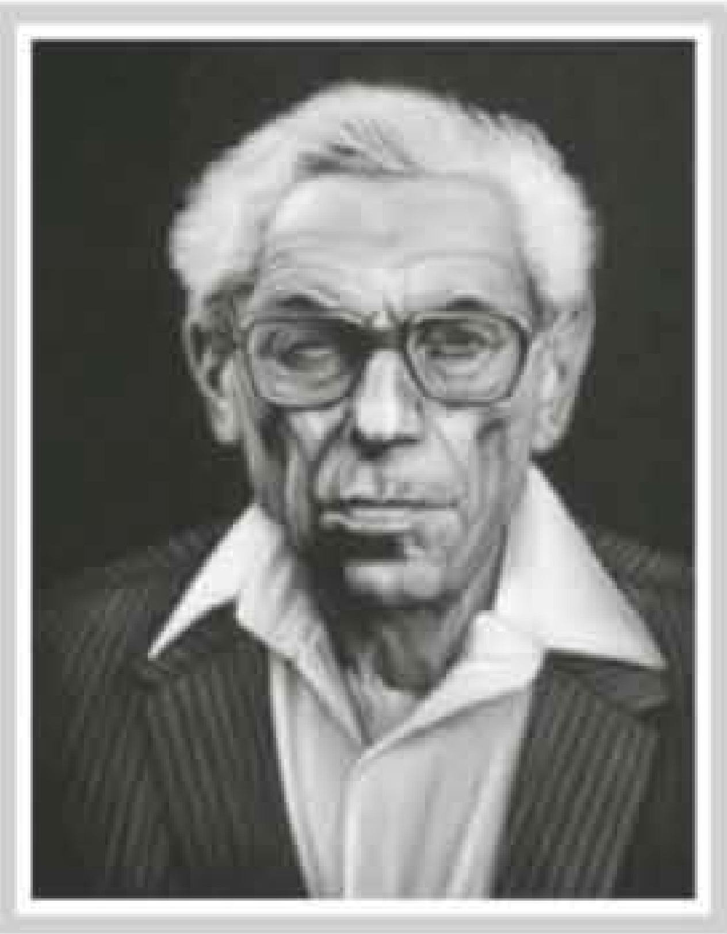
The young man who discussed mathematical concepts with Erdős on the beach was Krishnaswami Alladi, the son of Alladi Ramakrishnan, founder of the Institute of Mathematical Sciences (Matscience) in Chennai. While Ramakrishnan had invited many a brilliant scientist including Nobel laureates to the southern capital to talk about their work, Erdős had come mainly to speak to Krishnaswami.

Nomadic genius

Krishnaswami, who is now a professor of mathematics at the University of Florida, Gainesville, recalls how he had contacted the nomadic genius. As a B.Sc. student, working on an independent project on number theory, he had made some discoveries and had come up with questions that no one around him knew the answers to. He spoke to many people in the field in India and abroad. One name that came up a lot was Erdős, but no one knew his precise whereabouts. So, he was advised to write to him c/o The Hungarian Academy of Sciences. Within a month, Erdős responded, saying he was going to speak at a symposium at the Indian Statistical Institute in Calcutta later that year — could they meet there?

Krishnaswami's paper had also been accepted for the symposium, but he had his college half-yearly examinations and couldn't possibly travel to Calcutta that week. So, his father, an invited speaker at the symposium, offered to present the paper. Krishnaswami recounts that at the end of the presentation, Erdős came up to his father and said, "I am very pleased to meet you, but I'd be much happier to meet your son." Erdős was leaving for Australia the next week but was happy to re-route his trip via Madras.

Generosity and passion



This speaks volumes of his generosity and his passion to encourage young mathematicians, says Krishnaswami. When he went to the airport to receive the visitor, he recalls being nervous but Erdős broke the ice by reciting this poem.

This is the city of Madras / The home
of the curry and the dal
Where Iyers speak only to Iyengars/
And Iyengars speak only to God.

Erdős said he'd modelled it after this ditty about the privileged New England families famously known as the 'Boston Brahmins'.

This is good old Boston/ The home of
the bean and the cod
Where the Lowells speak to the
Cabots/ And the Cabots speak only
to God.

Erdős also met the Governor of Tamil Nadu, K.K. Shah who was amazed that such a frail-looking man could withstand the rigours of international travel. The minute Erdős heard of the governor's fund for scholarships to high-school students who excelled in mathematics, he reached into his pocket and pulled out a sheaf of rupee notes — the amount he was given for lectures in Madras — and donated it to the fund.

Before he left, Erdős asked Krishnaswami about his plans for graduate school. And right there, he wrote him a letter of recommendation to University of California, Los Angeles (UCLA). Ernst Straus, who had worked with Albert Einstein on relativity, would be his thesis advisor. It was a perfect fit, says Krishnaswami. In the fall of '75, he started graduate studies with a full fellowship at UCLA.

On another trip to Chennai, when Erdős heard of mathematician Srinivasa Ramanujan's destitute widow, he wrote her a cheque. This quirky genius gave away most of his earnings.

He had few possessions and travelled the world, solving and posing research problems with collaborators.

Mathematicians of the 20th century like to brag about their 'Erdős number' or their collaborative distance to him. Every co-author of Erdős's has the coveted Erdős number of one. Ramakrishnan has written five papers with Erdős, the first of which was based on the topic they discussed that afternoon on the Marina Beach. Publishing with Erdős's co-author would give a person an Erdős number of two. Ramanujan had an Erdős number of two. Though they never met, both had co-authored papers with the Cambridge mathematician G.H. Hardy. Einstein has an Erdős number of two via Straus. Erdős, who was at the centre of it all, has an Erdős number of zero.

The perfect end

In 1996, Erdős died of a heart attack aged 83, while at a conference in Poland. An obituary said he often mused about the perfect death. It would occur just after a lecture, when he had finished presenting a proof, and a cantankerous member of the audience would raise a hand to ask, "What about the general case?" He would respond: "I think I'll leave that to the next generation," and then keel over and die.

The ending was not so dramatic; and his legacy far more substantial. Through his life and work, he continues to inspire mathematicians to find elegant solutions to long-standing puzzles and problems.

(Alladi Krishnaswami, who edited his father's memoir, The Alladi Diary, is currently writing his own memoir titled Mathematics: People, Personalities and the Profession to be published by World Scientific.)

Courtesy : The Hindu

MATHS MUSING

SOLUTION SET-199

1. **(c) :** Equation of tangent to the parabola $y^2 = 4x$ having slope m is $y = mx + \frac{1}{m}$.

Equation of tangent to the circle $(x-1)^2 + (y+2)^2 = 16$ having slope m is $(y+2) = m(x-1) + 4\sqrt{1+m^2}$.

Distance between tangents (d)

$$= \left|\frac{4\sqrt{1+m^2} - m - 2 - 1/m}{\sqrt{1+m^2}}\right| = \left|4 - \frac{2}{\sqrt{1+m^2}} - \frac{\sqrt{m^2+1}}{m}\right|$$

Now, if $m > 0$, then $d < 4$ or $y = x + 1$.

2. **(c) :** Let T_1 = Period of $\sin\{\sin(nx)\} = \frac{2\pi}{n}$ and

T_2 = Period of $\tan\left(\frac{x}{n}\right) = \frac{\pi}{1/n} = \pi n$

$\therefore$ Period of $f(x)$ = L.C.M. of $\{T_1, T_2\}$

$$= \text{L.C.M. of } \left(\frac{2\pi}{n}, \pi n\right) = \text{L.C.M. of} \left\{\frac{2n\pi}{n^2}, \frac{2n\pi}{2}\right\}$$

$$= \frac{2n\pi}{1} = 6\pi \quad \text{(Given)} \quad \therefore n = 3$$

3. **(a) :** $17^{1983} + 11^{1983} - 7^{1983} = (17^{1983} - 7^{1983}) + 11^{1983}$

$= (17-7)(17^{1982} + 17^{1981}\cdot 7^1 + 17^{1980}\cdot 7^2 + \ldots + 7^{1982}) + 11^{1983}$

$\therefore$ Last digit in $17^{1983} - 7^{1983}$ is 0 and last digit in 11^{1983} is 1

$\therefore$ Unit digit of $17^{1983} + 11^{1983} - 7^{1983}$ is $0 + 1 = 1$.

4. **(a) :** $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$= 5(\hat{i} + 2\hat{j} + 2\hat{k}) - 6(\hat{i} + \hat{j} + 2\hat{k})$

$\Rightarrow (1+\alpha)\hat{i} + \beta(1+\alpha)\hat{j} + \gamma(1+\alpha)(1+\beta)\hat{k} = -\hat{i} + 4\hat{j} - 2\hat{k}$

$\Rightarrow 1 + \alpha = -1, \beta = -4$ and $\gamma(-1)(-3) = -2 \Rightarrow \gamma = -2/3$

5. **(a) :** $$\lim_{x\to 0^+}\left[\left(1-e^x\right)\frac{\sin x}{|x|}\right] = \lim_{x\to 0^+}\left[(0^-)\frac{\sin x}{x}\right]$$

$$= [0^-] = -1$$

$$\lim_{x\to 0^-}\left[\left(1-e^x\right)\frac{\sin x}{|x|}\right] = \lim_{x\to 0^-}\left[(0^+)\frac{\sin x}{-x}\right] = [0^-] = -1$$

Hence, $\lim_{x\to 0}\left[\left(1-e^x\right)\frac{\sin x}{|x|}\right] = -1$.

6. **(a, c) :** Let p_1, p_2 be the chances of happening of the first and second events, respectively.

Then according to the given conditions, we have

$$p_1 = p_2^2 \text{ and } \frac{1-p_1}{p_1} = \left(\frac{1-p_2}{p_2}\right)^3$$

Hence, $\frac{1-p_2^2}{p_2^2} = \left(\frac{1-p_2}{p_2}\right)^3 \Rightarrow p_2(1+p_2) = (1-p_2)^2$

$\Rightarrow 3p_2 = 1 \Rightarrow p_2 = 1/3$ and $\Rightarrow p_1 = 1/9$.

(7-8) : 7. (d) 8. (b)

We have, $a + b + c = 25$ ...(i)

$2a = b + 2$ ...(ii) $c^2 = 18b$ ...(iii)

From (i) and (ii), we have $b = 16 - \frac{2c}{3}$

From (iii), $c^2 = 18\left(16 - \frac{2c}{3}\right) \Rightarrow c^2 + 12c - 288 = 0$

$\Rightarrow (c-12)(c+24) = 0$

Now, $c = -24$ is not possible since it does not lie between 2 and 18. So, $c = 12$.

Thus, $a = 5, b = 8$ and $c = 12$. Hence, $abc = 5 \times 8 \times 12 = 480$.

If a, b, c are roots of the equation $x^3 + qx^2 + rx + s = 0$, then the sum of product of roots taken two at a time is $r = 5 \times 8 + 5 \times 12 + 8 \times 12 = 196$.

9. **(6) :** Let $z = x + iy \Rightarrow \bar{z} = x - iy$

Now, $E = z\bar{z} + (z-3)(\bar{z}-3) + (z-6i)(\bar{z}+6i)$

$= 3z\bar{z} - 3(z+\bar{z}) + 9 + 6(z-\bar{z})i + 36$

$= 3(x^2+y^2) - 6x - 12y + 45$

$= 3[x^2 + y^2 - 2x - 4y + 15] = 3[(x-1)^2 + (y-2)^2 + 10]$

$\therefore E_{\min} = 30$ when $x = 1$ and $y = 2$.

10. **(b) : P → 2, 4, 5; Q → 1, 2; R → 3, 5**

(P) $\int_1^4 |x-3|\,dx = \int_1^3 (3-x)dx + \int_3^4 (x-3)dx$

$$= \left\{3x - \frac{x^2}{2}\right\}_1^3 + \left\{\frac{x^2}{2} - 3x\right\}_3^4$$

$$= \left\{\left(9 - \frac{9}{2}\right) - \left(3 - \frac{1}{2}\right)\right\} + \left\{(8-12) - \left(\frac{9}{2} - 9\right)\right\} = 5/2$$

$\Rightarrow 2a + b = 5/2$

$\therefore a = 1, b = 1/2; a = 2, b = -3/2; a = 1/2, b = 3/2$

(Q) $\int_{-1}^1 \frac{|x|}{x}dx = \int_{-1}^0 (-1)\,dx + \int_0^1 1dx$

$= -(0+1) + (1-0) = 0 \Rightarrow a - 2b = 0$ or $a = 2b$

$\therefore a = 3, b = 3/2; a = 1, b = 1/2$

(R) $\int_0^{2\pi} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$

$= -\left[\cos x\right]_0^{\pi} + \left[\cos x\right]_{\pi}^{2\pi} = -(-1-1) + (1-(-1)) = 4$

$\Rightarrow 5a + b = 4 \therefore a = 1, b = -1; a = 1/2, b = 3/2$

◆ ◆

mtG

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